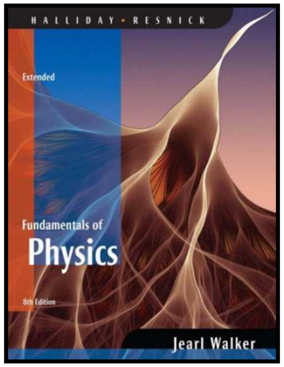
University Physics II



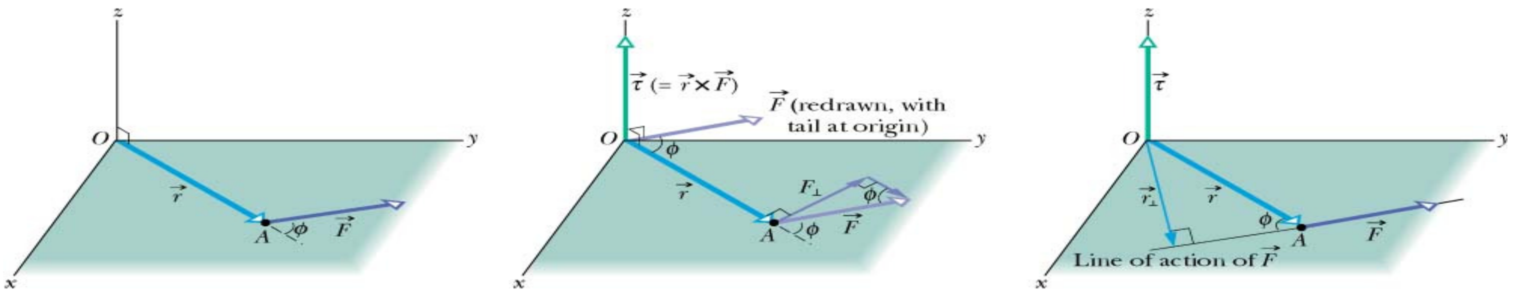
Week 3 : Day 2

Outline

- **Torque Revisited**
  - Two definitions for torque
- **Angular momentum**
  - The definition of angular momentum
  - Relating angular momentum and torque
  - Conservation of angular momentum
- **Activity – Angular Momentum Introduction**
  - Rotating platform – changing angular momentum
  - Stationary platform – conserving angular momentum
  - The Gyroscope – using angular momentum

Torque - Revisited

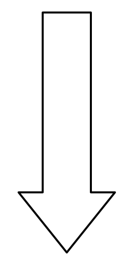
In Chapter 10 we defined the torque τ of a rigid body rotating about a fixed axis with each particle in the body moving on a circular path. We now expand the definition of torque so that it can describe the motion of a particle that moves along any path relative to a fixed point. If $\vec{r}$ is the position vector of a particle on which a force $\vec{F}$ is acting, the torque $\vec{\tau}$ is defined as $\vec{\tau} = \vec{r} \times \vec{F}$.



In the example shown in the figure both $\vec{r}$ and $\vec{F}$ lie in the xy-plane. Using the right-hand rule we can see that the direction of $\vec{\tau}$ is along the z-axis.
The magnitude of the torque vector $\tau = rF \sin\phi$, where ϕ is the angle between $\vec{r}$ and $\vec{F}$.

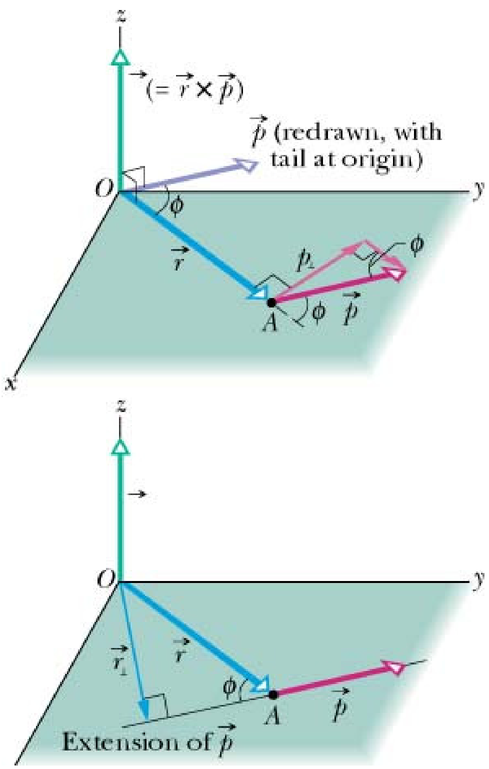
Two Views of Torque

$$\vec{\tau} = I\vec{\alpha}$$



$$\vec{\tau} = \vec{r} \times \vec{F}$$

Defining Angular Momentum, $\vec{L}$



The counterpart of linear momentum $\vec{p} = m\vec{v}$ in rotational motion is a new vector known as angular momentum.

The new vector is defined as follows: $\vec{\ell} = \vec{r} \times \vec{p}$.

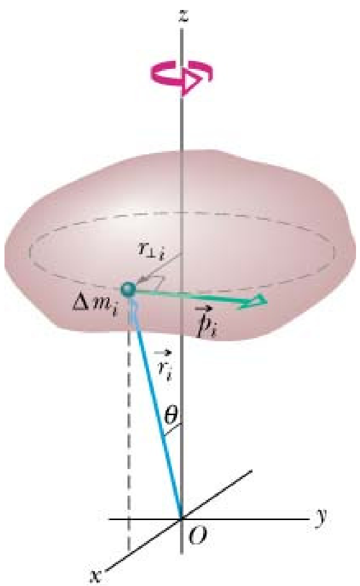
In the example shown in the figure both $\vec{r}$ and $\vec{p}$ lie in the xy-plane. Using the right-hand rule we can see that the direction of $\vec{\ell}$ is along the z-axis.

The magnitude of angular momentum $\ell = rmv\sin\phi$, where ϕ is the angle between $\vec{r}$ and $\vec{p}$. From triangle OAB we have: $r\sin\phi = r_\perp \rightarrow \ell = r_\perp mv$.

Note: Angular momentum depends on the choice of the origin O. If the origin is shifted, in general we get a different value of $\vec{\ell}$.

SI unit for angular momentum: $\text{kg}\cdot\text{m}^2/\text{s}$. Sometimes the equivalent $\text{J}\cdot\text{s}$ is used.

Angular Momentum of a Rigid Body



We take the z-axis to be the fixed rotation axis. We will determine the z-component of the net angular momentum. The body is divided into n elements of mass Δm_i that have a position vector $\vec{r}_i$. The angular momentum $\vec{\ell}_i$ of the ith element is $\vec{\ell}_i = \vec{r}_i \times \vec{p}_i$. Its magnitude is $\ell_i = r_i p_i (\sin 90°) = r_i \Delta m_i v_i$. The z-compoment ℓ_{iz} of ℓ_i is $\ell_{iz} = \ell_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i) = r_{\perp i} \Delta m_i v_i$.

The z-component of the angular momentum L_z is the sum:

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} r_{\perp i} \Delta m_i v_i = \sum_{i=1}^{n} r_{\perp i} \Delta m_i (\omega r_{i\perp}) = \omega \left(\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2 \right)$$

The sum $\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2$ is the rotational inertia I of the rigid body.

$$L_z = I\omega$$

Derive angular momentum from Newton's second Law as follows:

$$\vec{\tau} = \vec{\tau}$$

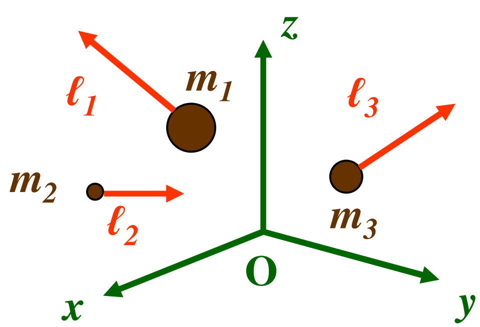
$$\Rightarrow \vec{r} \times \vec{F} = I\vec{\alpha}$$

$$\Rightarrow \vec{r} \times \frac{d}{dt}\vec{p} = I\frac{d}{dt}\vec{\omega}$$

$$\Rightarrow \vec{r} \times \vec{p} = I\vec{\omega}$$

$$\Rightarrow \vec{L} = I\vec{\omega}$$

Relating $\vec{\tau}$ and $\vec{L}$



We will now explore Newton's second law in angular form for a system of n particles that have angular momentum $\vec{\ell}_1, \vec{\ell}_2, \vec{\ell}_3, \ldots, \vec{\ell}_n$.

The angular momentum $\vec{L}$ of the system is $\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \ldots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i$.

The time derivative of the angular momentum is $\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt}$.

From Newton's Second Law we see that…

$$\vec{\tau} = I\vec{\alpha}$$

$$\Rightarrow \vec{\tau} = I\frac{d}{dt}\vec{\omega} = \frac{d}{dt}\left(I\vec{\omega}\right)$$

$$\Rightarrow \vec{\tau} = \frac{d}{dt}\vec{L}$$

If no net torque then $\vec{L}$ is a constant in time…

Conservation of $\vec{L}$

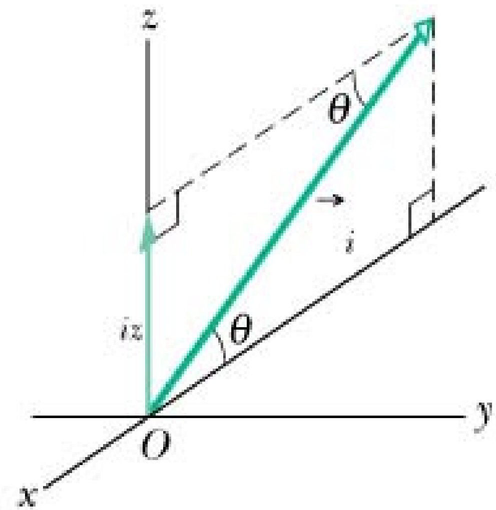
For any system of particles (including a rigid body) Newton's second law in angular form is $\dfrac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}$.

If the net external torque $\vec{\tau}_{\text{net}} = 0$ then we have: $\dfrac{d\vec{L}}{dt} = 0 \rightarrow$ $\vec{L} =$ a constant. This result is known as the law of the conservation of angular momentum. In words:

$$\begin{pmatrix}\text{Net angular momentum} \\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{Net angular momentum} \\ \text{at some later time } t_f\end{pmatrix}$$

In equation form: $$\vec{L}_i = \vec{L}_f$$

Note: If the component of the external torque along a certain axis is equal to zero, then the component of the angular momentum of the system along this axis cannot change.



RIT

Activity – Angular Momentum Introduction

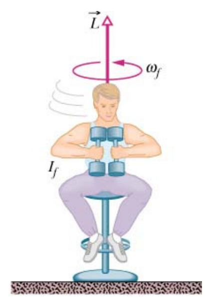
- **Changing angular momentum**
  - See how changing I changes your rotation rate ω…

$$L_z = I\omega$$

  - Conservation of momentum tells us that…

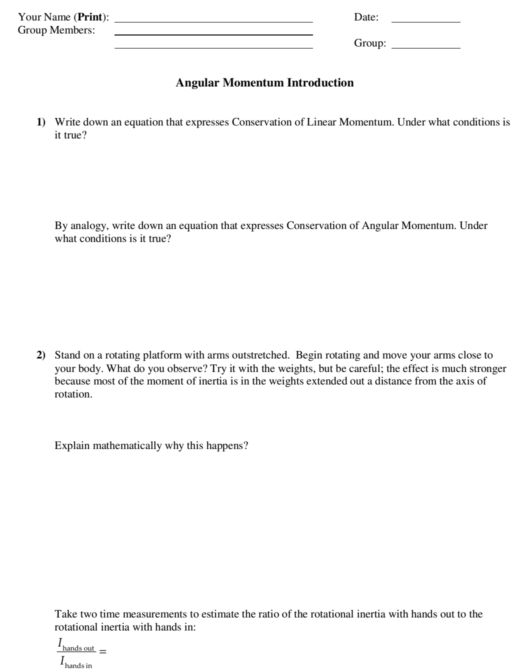
Platform is Initially Rotating…

$$\vec{L}_i = \vec{L}_f$$



  - When I changes what happens to ω?

Activity



Your Name (**Print**): ____________________ Date: __________
Group Members: ____________________ Group: __________

Angular Momentum Introduction

1) Write down an equation that expresses Conservation of Linear Momentum. Under what conditions is it true?

By analogy, write down an equation that expresses Conservation of Angular Momentum. Under what conditions is it true?

2) Stand on a rotating platform with arms outstretched. Begin rotating and move your arms close to your body. What do you observe? Try it with the weights, but be careful; the effect is much stronger because most of the moment of inertia is in the weights extended out a distance from the axis of rotation.

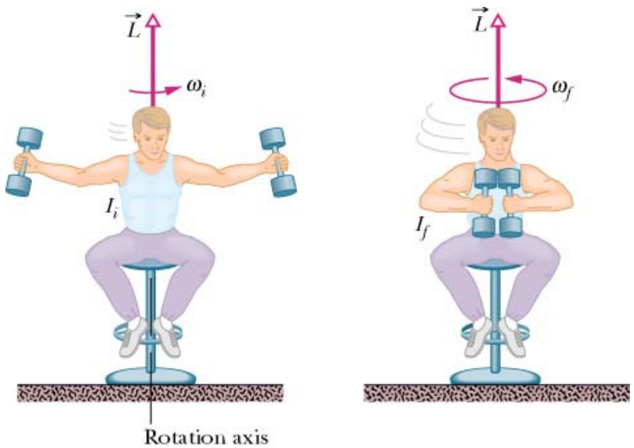
Explain mathematically why this happens?

Take two time measurements to estimate the ratio of the rotational inertia with hands out to the rotational inertia with hands in:

$$\frac{I_{\text{hands out}}}{I_{\text{hands in}}} =$$

RIT

Explaining what we observe...



The student, who has been set into rotation at an initial angular speed ω_i, holds two dumbbells in his outstretched hands. His angular momentum vector $\vec{L}$ lies along the rotation axis, pointing upward.

The student then pulls in his hands as shown in fig. b. This action reduces the rotational inertia from an initial value I_i to a smaller final value I_f.

No net external torque acts on the student-stool system. Thus the angular momentum of the system remains unchanged.

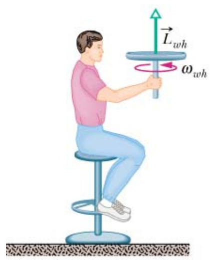
Angular momentum at t_i: $L_i = I_i\omega_i$. Angular momentum at t_f: $L_f = I_f\omega_f$.

$$L_i = L_f \rightarrow I_i\omega_i = I_f\omega_f \rightarrow \omega_f = \frac{I_i}{I_f}\omega_i. \quad \text{Since } I_f < I_i \rightarrow \frac{I_i}{I_f} > 1 \rightarrow \quad \omega_f > \omega_i.$$

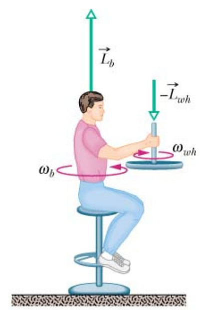
Activity – Angular Momentum Introduction

- **Conserving angular momentum**
  - Observe what happens when momentum is red-directed...
  - Conservation of angular momentum tells us…

Platform is Initially Stationary…

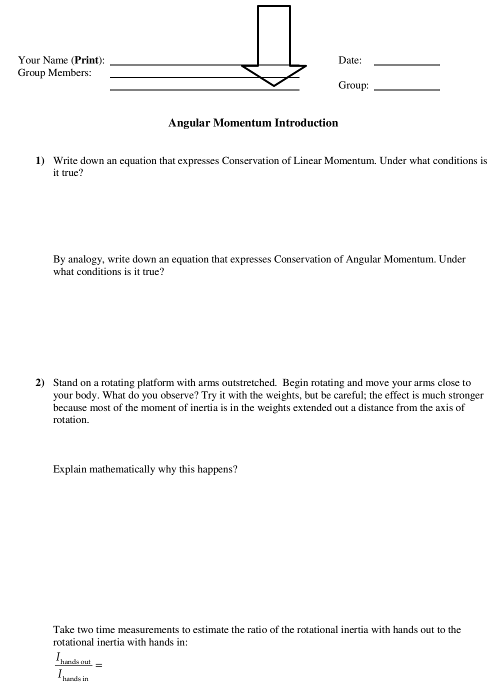


$$\vec{L}_i = \vec{L}_f$$



  - What is initial and final angular momentum of the system?
    - *Consider the direction of L for the wheel and the platform before and after…*

Activity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Explain mathematically why this happens?

Take two time measurements to estimate the ratio of the rotational inertia with hands out to the rotational inertia with hands in:

$\frac{I_{\text{hands out}}}{I_{\text{hands in}}} =$

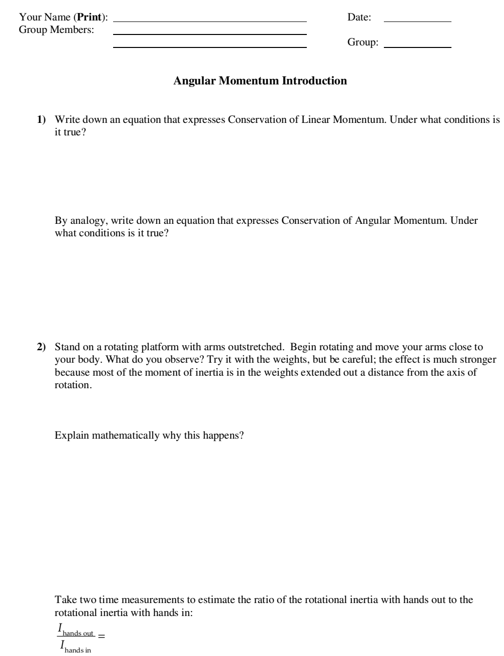
RIT

Activity – Angular Momentum Introduction

- **Using angular momentum**
  - Start the gyroscope spinning…

  - Observe the response of the gyroscope when you perturb it by pushing from various directions (right, left, up, down)
    - *Which way does the <u>spinning</u> gyroscope want to move?*

Activity



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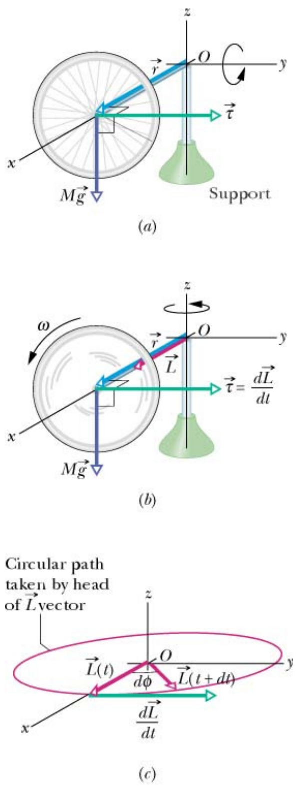
RIT

Uses for Gyroscope Motion

- **Precession of gyroscope**
  - Easily quantified (The axle of the spinning wheel defines the spin axis)
  - Used in many applications (used to construct gyrocompasses which complement or replace magnetic compasses)



The torque is applied perpendicular to the axis of rotation, and therefore perpendicular to the angular momentum

$$\vec{\tau} = \frac{d}{dt}\vec{L} \equiv \vec{\Omega}_p \times \vec{L}$$

Precession Angle per unit Time

PATENTS

- U.S. Patent 839,161, "Steering apparatus for automobile torpedoes".
- U.S. Patent 795,045, "Gyroscopic control apparatus".
- U.S. Patent 785,587, "Mechanical speed governor".
- U.S. Patent 785,425, "Steering mechanism for torpedoes".
- U.S. Patent 751,888, "Governing mechanism for turbines".
- U.S. Patent 738,823, "Electrical apparatus".
- U.S. Patent 730,613, "Meter".
- U.S. Patent 662,484, "Electric top for gyroscopes".
- U.S. Patent 648,878, "Gyroscope for torpedo steering mechanism".
- U.S. Patent 642,704, "Roller bearing car wheel".
- U.S. Patent 484,960, "Gyroscopic top".
- U.S. Patent 461,948, "Gyroscope or revolving toy".
- U.S. Patent 365,530, "Lumber cart".
- U.S. Patent 312,692, "Vehicle wheel".
- U.S. Patent 220,867, "Engine-governor and speed-regulator".
- U.S. Patent 162,446, "Governor for steam engine".
- U.S. Patent 34,298, "Levelling instrument".

See Wikipedia entry for more:

http://en.wikipedia.org/wiki/Gyroscope