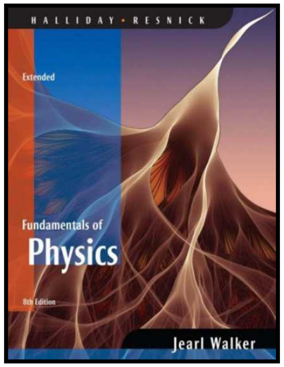
Workshop Physics 1017 - 312

University Physics II

Week 4 : Day 1



Outline

- **Direct Calculation of Moments of Inertia**
  - Integration methods
  - Integration examples
    - *One-dimensional (ID) Integration*
    - *Two-dimensional (2D) Integration*
    - *Three-dimensional (3D) Integration*
  - Activity – Rotational Inertia Integration

- **Torque and Vector Directions**
  - The Vector Cross Product
  - Activity – Vector Cross Products

- **The Atwood Machine**
  - Free-Body Diagram
  - Solutions and Simulations

Integration Methods

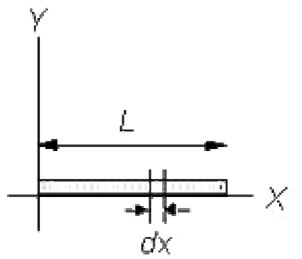
- Draw a <u>diagram</u> showing the object and the axis of rotation
- Choose an appropriate <u>coordinate system</u>
- Pick a "slice" (or piece), *dm*, and perpendicular distance *r*
  - Show the distance *r* on your diagram
- Write an expression for *dm* in terms of a small element
  - In 1-D problems $dm = \lambda\, dx$
  - In 2-D problems $dm = \sigma\, dxdy$
  - In 3-D problems $dm = \rho\, dxdydz$
- Perform the integral

$$I = \int r_{\perp}^{2}\, dm$$

Calculating Moments - ID

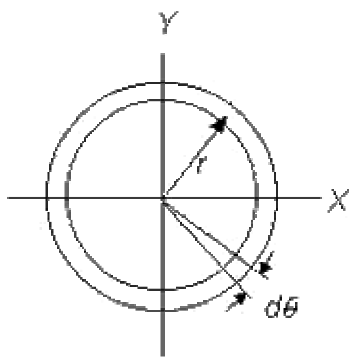
- Thin Rod

$$I_y = \int_0^L r_\perp^2 dm = \int_0^L x^2 \left(\underbrace{\frac{m}{L}}_{\lambda} \underbrace{dx}_{dl} \right) = \frac{m}{L} \frac{x^3}{3} \Bigg|_0^L = \frac{1}{3} mL^2$$



- Circular Ring

$$I_z = \int_0^{2\pi} r_\perp^2 dm = \int_0^{2\pi} r^2 \left(\underbrace{\frac{m}{2\pi r}}_{\lambda} \underbrace{r d\theta}_{dl} \right) = \frac{mr^2}{2\pi} \int_0^{2\pi} d\theta = mr^2$$



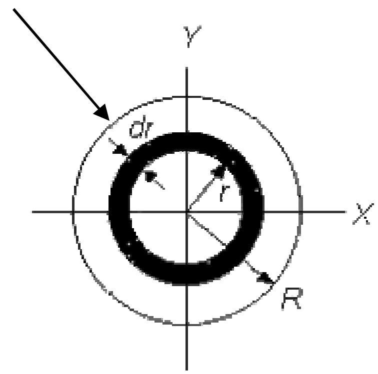
Calculating Moments - 2D

- Circular Disk

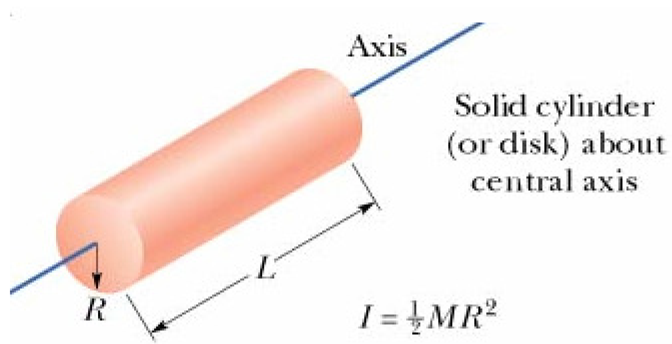
Note that the *mass element*, ***dm*** is 2D…

$$I_z = \int r_\perp^2 dm = \int_0^R \int_0^{2\pi} r^2 \overbrace{\underbrace{\left(\frac{M}{\pi R^2}\right)}_{\sigma} \underbrace{(r d\theta) dr}_{\text{Polar Coordinates}}}^{dm}$$

$$= \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2} \left. \frac{r^4}{4} \right|_0^R = \frac{1}{2} MR^2$$



Why doesn't the third dimension matter here?



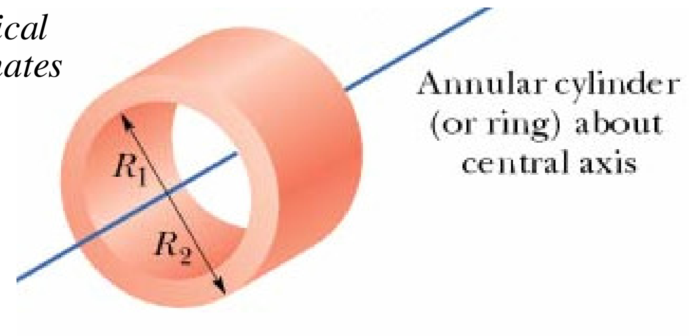
Calculating Moments - 3D

- **Massive Ring**

Note that the *mass element*, $\mathbf{dm}$ is 3D…

$$I_z = \int r_\perp^2 dm = \int_{R_1}^{R_2}\int_0^{2\pi}\int_0^L r^2 \underbrace{\overbrace{\left(\frac{M}{\pi\left(R_2^2 - R_1^2\right)L}\right)}_{\rho}\underbrace{(rd\theta)drdz}_{\text{Cylindrical Coordinates}}}^{dm}$$

$$I_{RING} = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$$

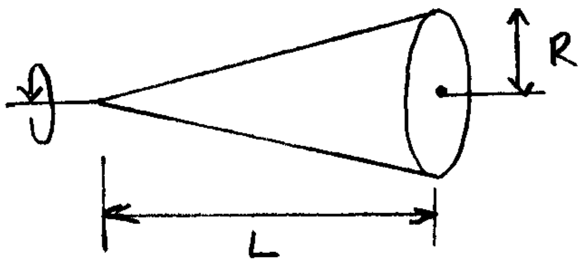


$$= \frac{2M}{\left(R_2^2 - R_1^2\right)}\int_{R_1}^{R_2} r^3 dr = \frac{2M}{\left(R_2^2 - R_1^2\right)}\left.\frac{r^4}{4}\right|_{R_1}^{R_2}$$

$$= \frac{1}{2}M\,\frac{R_2^4 - R_1^4}{\left(R_2^2 - R_1^2\right)} = \frac{1}{2}M\,\frac{\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)}{\left(R_2^2 - R_1^2\right)} = \frac{1}{2}M\left(R_2^2 + R_1^2\right)$$

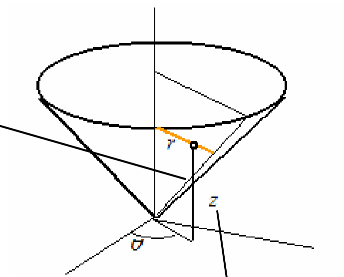
Example - Inertia of a Cone

- **Uniform mass density**
  - Start with definition
  - Need volume of cone



$$I_z = \int r_\perp^2 dm = \rho \int r^2 dV = \frac{M}{\frac{1}{3}\pi R^2 L} \int_0^L dz \int_0^{2\pi} d\varphi \int_0^{\frac{R}{L}z} dr r^3$$

$$\Rightarrow \frac{6M}{R^2 L} \int_0^L dz \left(\frac{r^4}{4} \right) \Bigg|_0^{\frac{R}{L}z} = \frac{3M}{2R^2 L} \int_0^L dz \frac{R^4}{L^4} z^4 = \frac{3}{10} MR^2$$



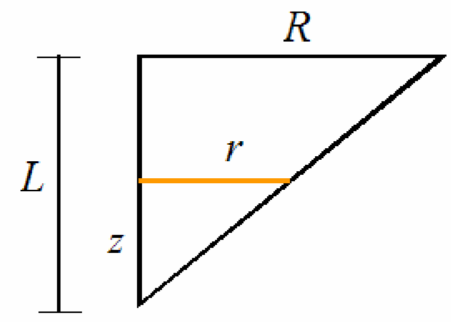
Equation of outer surface is a line…

Finding the Volume of a Cone

- **Uniform mass density**
  - Start with definition

$$V = \int dV = \int_0^L dz \int_0^{2\pi} d\varphi \int_0^{\frac{R}{L}z} dr\, r = 2\pi \int_0^L dz \left.\frac{r^2}{2}\right|_0^{\frac{R}{L}z}$$

$$= \pi \frac{R^2}{L^2} \int_0^L z^2 dz = \pi \frac{R^2}{L^2} \left.\frac{z^3}{3}\right|_0^L = \frac{1}{3}\boxed{\pi R^2 L}$$

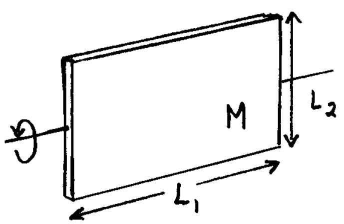


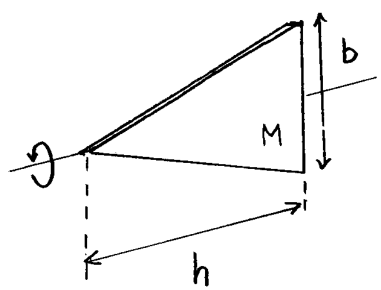
$$\frac{z}{L} = \frac{r}{R}$$

Use similar triangles to get equation of surface…

Activity – 2D/3D Integration

- **Use definition**
  - $$I = \int r_{\perp}^{2}\, dm$$
    - *Assume thickness is negligible…*
  - No tables!





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Group Members: ____________________

____________________ Group: __________

Rotational Inertia – Integration
(2D and 3D)

The rotational inertia (or moment of inertia) for a continuous mass distribution is defined as

$$I = \int_{\text{rigid body}} dI = \int_{\text{rigid body}} r^2\, dm$$

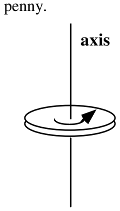
where r is the perpendicular distance of the mass dm from the axis of rotation.

The method to use is:

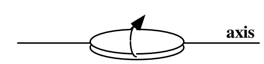
- draw a diagram showing the object and the axis of rotation
- choose a coordinate system that you think will be best to use
- pick a "slice" (or piece), dm, a perpendicular distance r from the axis of rotation; the piece should not be at the middle or the end of the object; show the distance r on your diagram
- write an expression for dm in terms of a small element of whatever coordinate(s) you have chosen
  - $dm = \sigma\, dA$ for 2-dimensional objects where σ is the mass/area and dA is the area of dm; dA will have to be expressed in terms of your spatial coordinates, for example, in Cartesian coordinates $dA = dx{\cdot}dy$)
  - $dm = \rho\, dV$ for 3D objects where ρ is the mass/volume and dV is the volume of dm; dV will have to be expressed in terms of your spatial coordinates, for example, in Cartesian coordinates $dV = dx{\cdot}dy{\cdot}dz$)
- substitute for dI in the integral definition
- do the integral.

1. Calculate I_{com} for a penny. Treat the penny as a flat, circular disk of mass M and radius R with a uniform mass distribution.
   a) Choose the axis to be perpendicular to the plane of the penny.
   b) Choose the axis to be in the plane of the penny.
   c) For both cases, also calculate the rotational inertia about a parallel axis passing through the edge of the penny.

(a)



(b)



HINT: Cut the disk up into small segments for which all of the mass is at the same distance from the axis. Then decide over what variable you have to integrate.

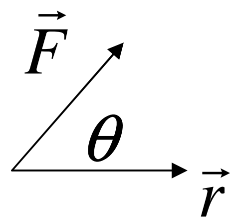
Torque - Vector Direction

- To determine the direction of a rotation it is sometime helpful to introduce the *unit vector* notation for the different directions:
  - $\hat{i}$ - Direction along X-Axis
  - $\hat{j}$ - Direction along Y-Axis
  - $\hat{k}$ - Direction along Z-Axis

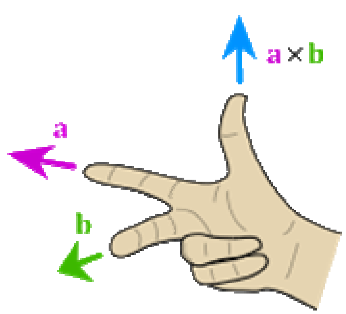
Magnitude of Torque

$$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}$$

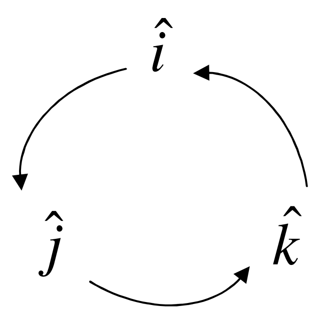
$$\Rightarrow rF \sin\theta = I\alpha$$



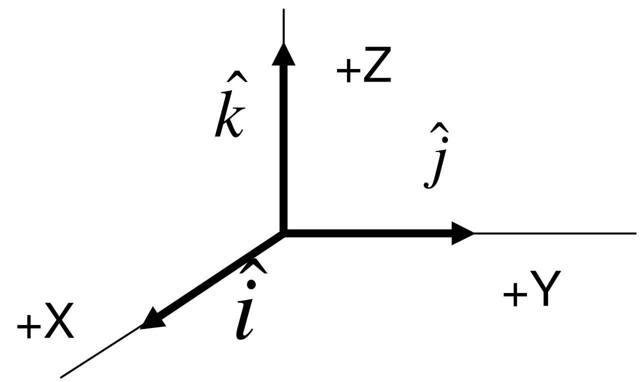
Right-Hand Rule



Vectors Permute



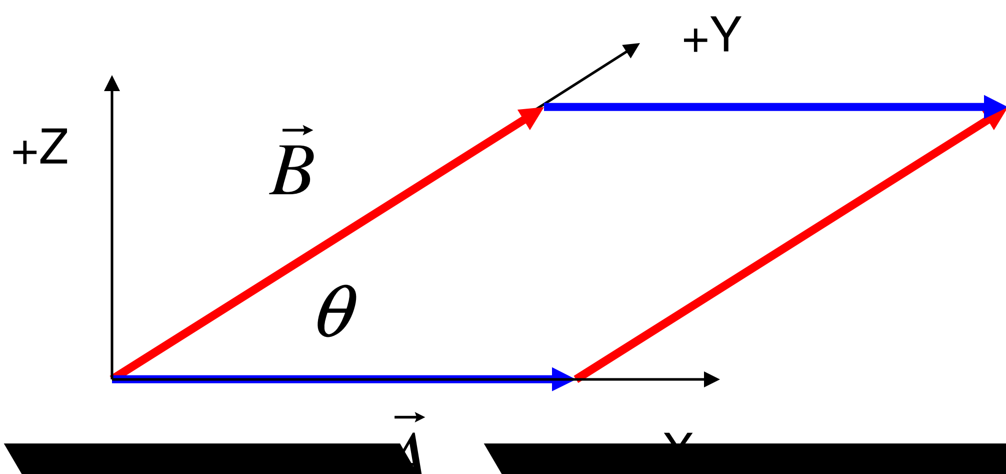
Right-Hand Coordinates



Activity - Vector Cross Product

- **_Vector Cross Product_**
  - Important in many areas of Physics

$$\left\|\vec{A}\times\vec{B}\right\| = \|A\|\|B\|\sin\theta$$



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Group Members: ____________________
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Vector Cross Product Practice

Evaluate the following cross products. There are three common ways to calculate these, all of which are equivalent and basically do the same thing;

a) Using the RH rule on the unit vectors.
b) Putting the components into the cyclical formula.
c) Using the determinant method.

Have each person in your group use a different one of the above techniques for the first question, and then rotate your choice of techniques for the next ones.

1) $(2\text{ m})\hat{i}\ \text{x}\ (3\text{ N})\hat{j} =$

2) $[\,(2\text{ m})\hat{i} + (3\text{ m})\hat{j} - (2\text{ m})\hat{k}\,]\ \text{x}\ [\,(3\text{ N})\ \hat{i} - (4\text{ N})\hat{k}\,] =$

3) $[\,(3\text{ m})\hat{i} - (4\text{ m})\hat{k}\,]\ \text{x}\ [\,(2\text{ N})\hat{i} + (3\text{ N})\hat{j} - (2\text{N})\hat{k}\,] =$

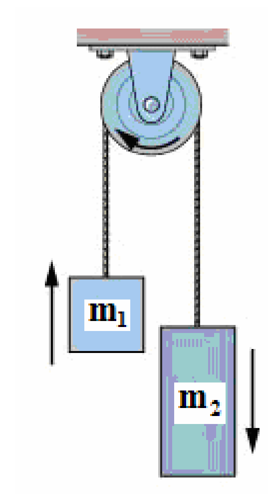
Main points to remember about the cross product

These are things that people very commonly make mistakes with on exams, so make sure you're clear about each one of these;

- ✓ Just like the dot product, the cross product has units. The units are the usual product of the units of each of the vectors in the cross product.
- ✓ Unlike the dot product, the order of the vectors matters in the cross product. Reverse the order and there's a sign flip in the result. Just think about the RH rule and you'll see that this is the case.
- ✓ The result of the cross product between two vectors is another vector, so don't just write a number down. Typically there will be three components, so show the proper unit vectors for each direction. (If a component happens to be zero, you can just leave it out, but all non-zero components must have the appropriate unit vectors next to them).

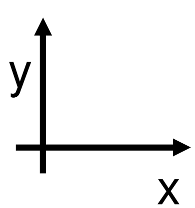
The Atwood Machine

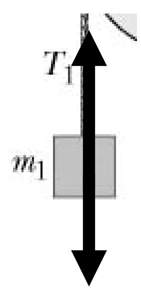
- **The Atwood Machine**
  - Invented in 1784 by Reverend George Atwood as a laboratory experiment to verify the mechanical laws of uniformly accelerated motion.
    (See Wikipedia entry for more...)

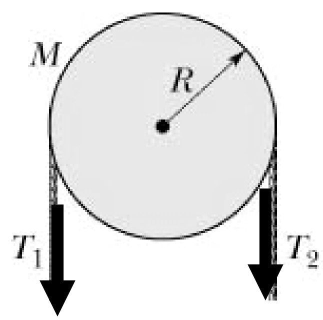


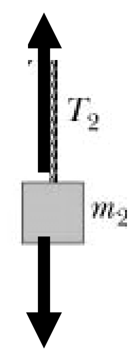
$$\sum \vec{F}, \sum \vec{\tau}$$

Free Body Diagram (FBD)









$$+T_1 - m_1 g = +m_1 a \qquad r\left(+T_1 - T_2\right) = I\left(-\alpha\right) \qquad +T_2 - m_2 g = -m_2 a$$

Atwood Solutions and Simulations

- **Solve the equations as follows:**

$$(1)\qquad +T_1 - m_1 g = +m_1 a \Rightarrow T_1 = m_1(g+a)$$

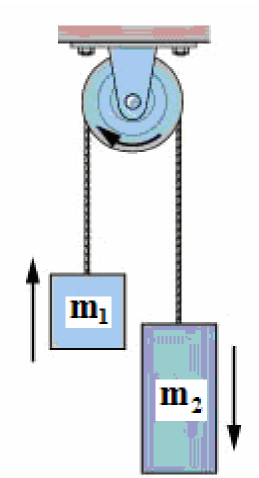
$$(2)\qquad +T_2 - m_2 g = -m_2 a \Rightarrow T_2 = m_2(g-a)$$

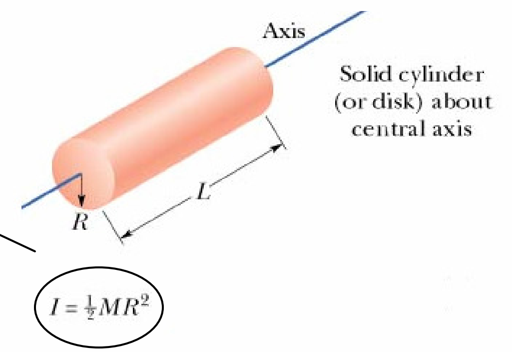
$$(3)\qquad r(+T_1 - T_2) = I(-\alpha)$$

$$\Rightarrow m_1(g+a) - m_2(g-a) = -\frac{I}{r}\left(\frac{a}{r}\right)$$

$$\Rightarrow (m_1 - m_2)g + (m_1 + m_2)a = -\frac{\frac{1}{2}Mr^2}{r^2}a$$

$$\Rightarrow \boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{1}{2}M}}$$





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