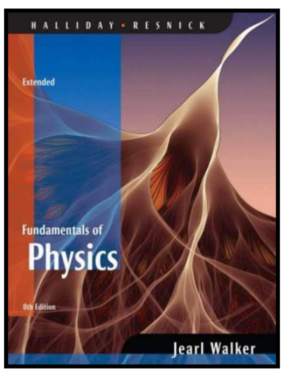
University Physics II

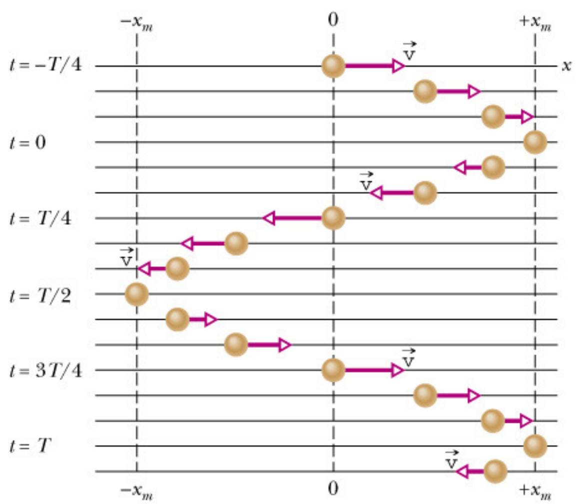
Week 5 : Day 1

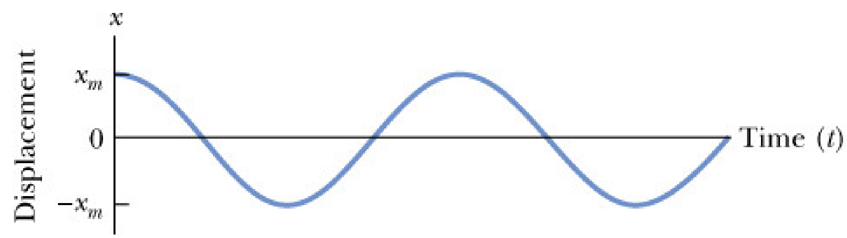


Outline

- **Harmonic Motion**
  - Amplitude, frequency, and phase
  - Velocity and acceleration
- **The SHO System**
  - The Standard SHO System
  - The Vertical SHO System
- **SHO Examples**
  - Torsion pendulum
  - Simple pendulum
    - *The small angle approximation*
    - *Conservation of angular momentum*
- **SHO Energy**
  - Kinetic and Potential Energy
  - Phase Relations
    - *Coupled oscillators*
- **Activities**
  - Aspects of Simple Harmonic Motion (in class)
  - The Torsion Pendulum (take home)

What is harmonic motion?





Simple Harmonic Motion (SHM)

In fig. *a* we show snapshots of a simple oscillating system.

The motion is periodic, i.e., it repeats in time. The time needed to complete one repetition is known as the period (symbol T, units: s). The number of repetitions per unit time is called the frequency (symbol f, unit hertz), $f = \dfrac{1}{T}$.

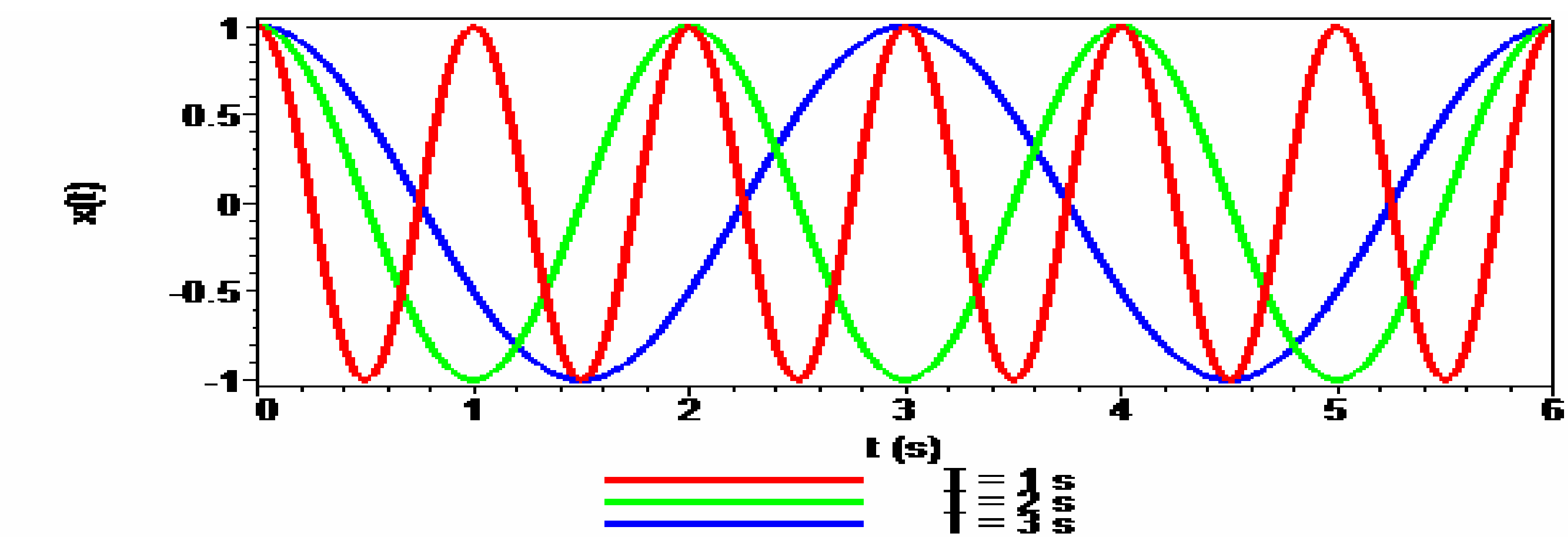
The displacement of the particle is given by the equation $x(t) = x_m \cos(\omega t + \phi)$.

Parameters of harmonic motion

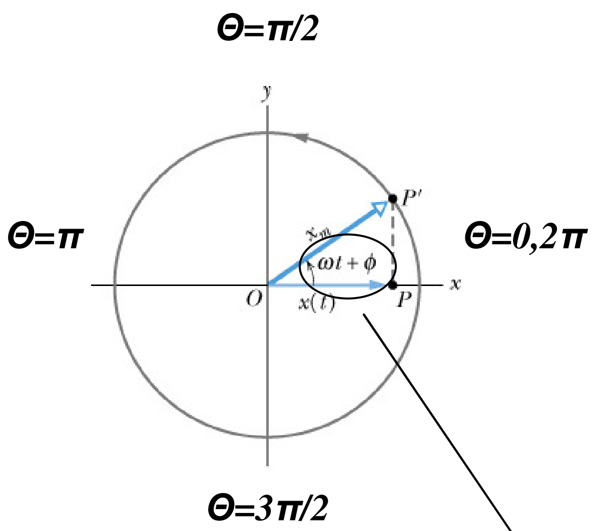
$$x(t) = x_m \cos(\omega t + \phi)$$

The quantity x_m is called the amplitude of the motion. It gives the maximum possible displacement of the oscillating object. The quantity ω is called the angular frequency of the oscillator. It is given by the equation

$$\omega = 2\pi f = \frac{2\pi}{T}$$



Harmonic and Circular Motion



Consider an object moving on a circular path of radius x_m with a uniform speed v. If we project the position of the moving particle at point P' on the x-axis we get point P. The coordinate of P is $x(t) = x_m \cos(\omega t + \phi)$.

While point P' executes uniform circular motion, its projection P moves along the x-axis with simple harmonic motion.

The phase angle determines the starting position of the harmonic motion…

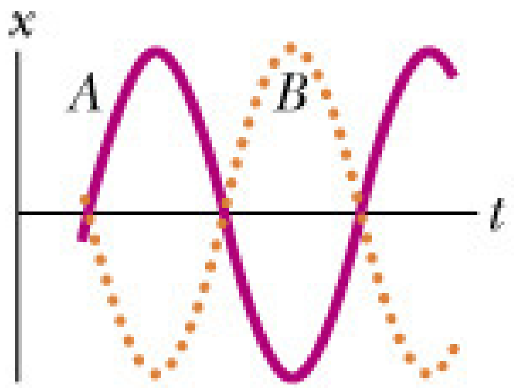
$$\theta(t) = \omega t + \phi \quad \Rightarrow \quad \theta(t=0) = \phi$$

Phase and Phase Angle

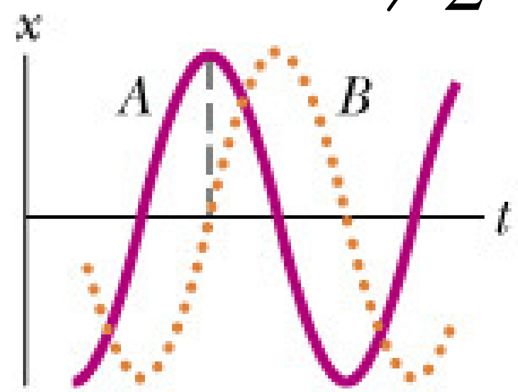
- **Phase**
  - Argument of the wave
- **Phase Angle**
  - Used to set the initial condition for one wave
  - Establish relationship between two waves

$$x(t) = \underbrace{x_m}_{Amplitude} \cos\Big(\underbrace{\overbrace{\omega}^{\text{Angular Frequency}} t + \overbrace{\phi}^{\text{Phase Angle}}}_{Phase}\Big)$$

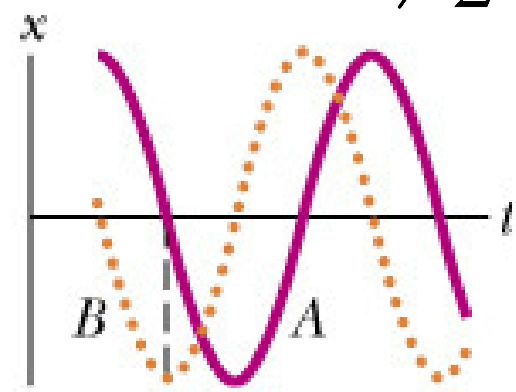
$$\phi_A - \phi_B = \pi$$



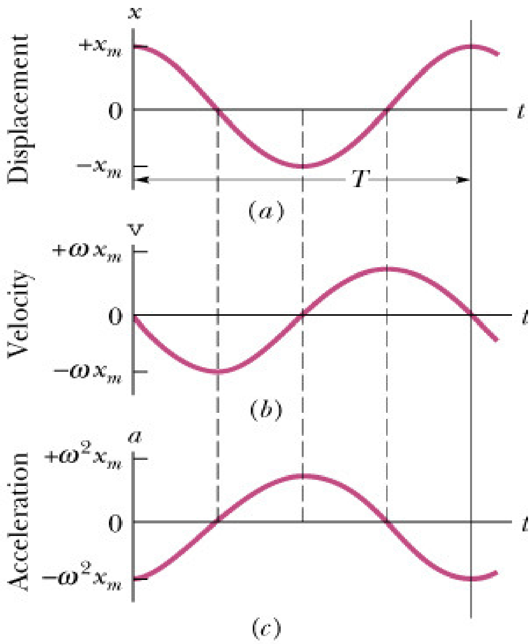
$$\phi_A - \phi_B = \pi/2$$



$$\phi_A - \phi_B = -\pi/2$$



Velocity and Acceleration



$x(t) = x_m \cos(\omega t + \phi)$ In fig. a $x(t)$ is plotted versus t for $\phi = 0$: $x(t) = x_m \cos \omega t$.

Velocity of SHM

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right] = -\omega x_m \sin(\omega t + \phi)$$

The quantity ωx_m is called the velocity amplitude v_m. It expresses the maximum possible value of $v(t)$.

In fig. b the velocity $v(t)$ is plotted versus t for $\phi = 0$: $v(t) = -\omega x_m \sin \omega t$.

Acceleration of SHM $$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[-\omega x_m \sin(\omega t + \phi)\right] = -\omega^2 x_m \cos \omega t = -\omega^2 x$$

The quantity $\omega^2 x_m$ is called the acceleration amplitude a_m. It expresses the maximum possible value of $a(t)$. In fig. c the acceleration $a(t)$ is plotted versus t for $\phi = 0$: $a(t) = -\omega^2 x_m \cos \omega t$.

The Basic SHO System

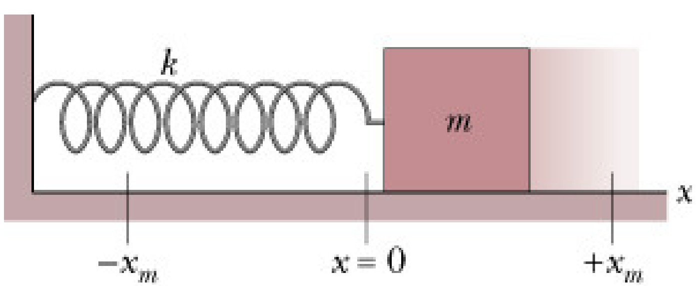
The Force Law for Simple Harmonic Motion

We saw that the acceleration of an object undergoing SHM is $a = -\omega^2 x$.

If we apply Newton's second law we get: $F = ma = -m\omega^2 x = -\left(m\omega^2\right)x.$

Simple harmonic motion occurs when the force acting on an object is proportional to the displacement but opposite in sign. The force can be written as $F = -Cx$ where C is a constant. If we compare the two expressions for F we have

$m\omega^2 = C \;\rightarrow$ and $T = 2\pi\sqrt{\frac{m}{C}}$



$$m\frac{d^2x}{dt^2} = -Cx$$

$$\Rightarrow \boxed{a \propto -x(t)}$$

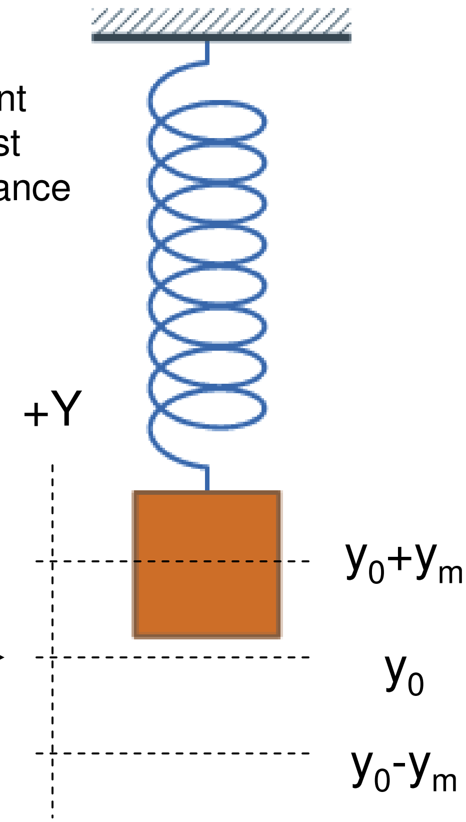
The Vertical SHO System

- Begin with Newton's law in (linear form)
  - $F = ma$
- Use Hooke's law for the force
  - $F = -k(y-y_0)$
- Construct the differential equation
  - $md^2y/dt^2+ky=0$
- Substitute the solution $y(t)=y_m cos(\omega t+\varphi)$*
  - $-\omega^2 y(t)+(k/m)y(t)=0$
- Solve for the angular frequency
  - $\omega^2 = k/m$
- Express the period of motion
  - $T = 2\pi\sqrt{m/k}$

y_0 is Amount spring must stretch to balance weight…

Choose $y_0=0$ as new equilibrium point for motion…

Vertical SHO System



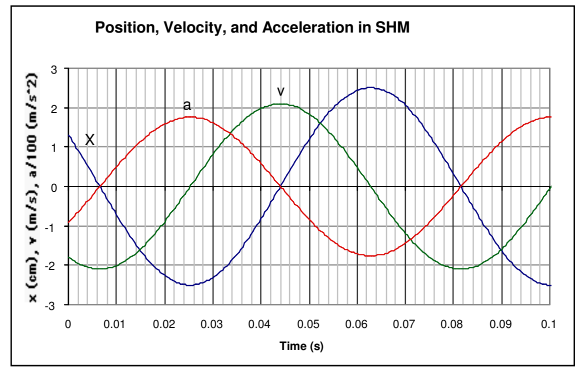
Activity – Aspects of SHM

- **Work through first three parts:**
  - Determining Parameters from Data Plots
  - What factors determine the period of oscillation? (Intuitive predictions)
  - Measurement of Periodic Parameters

Your Name (**Print**): ______________________ Date: ________
Group Members: ______________________ Group: ________

<u>**Aspects of Simple Harmonic Motion**</u>

A) Determining Parameters from Data Plots



Using the plots above, determine the following parameters for this simple harmonic motion:

Period, T ______________________

Frequency, f ______________________

Angular frequency, ω ______________________

Amplitude, x_m ______________________

Maximum speed, v_m ______________________

Maximum acceleration a_m ______________________

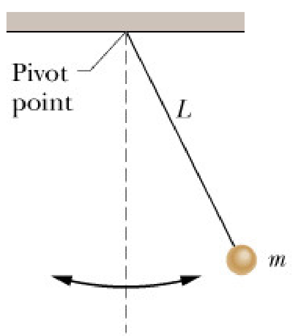
Initial Phase, ϕ ______________________

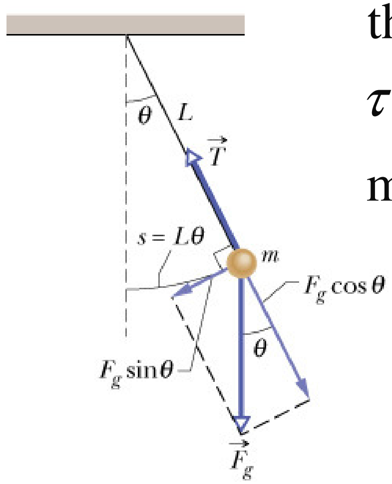
Suppose the plots describe a 50 g mass attached to a spring and set into simple harmonic motion. What are the

Spring Constant, k ______________________

Total Mechanical Energy, E ______________________

The Simple Pendulum





A simple pendulum consists of a particle of mass m suspended by a string of length L from a pivot point. If the mass is disturbed from its equilibrium position the net force acting on it is such that the system executes simple harmonic motion. There are two forces acting on m: the gravitational force and the tension from the string. The net torque of these forces is $\tau = -r_{\perp} F_g = -Lmg\sin\theta$. Here θ is the angle that the thread makes with the vertical.

$$\underbrace{I\alpha}_{\tau} = -Lmg\sin\theta$$

$$\Rightarrow \frac{d^2\theta(t)}{dt^2} + \underbrace{\left(\frac{Lmg}{I}\right)}_{constant}\sin\theta(t) = 0$$

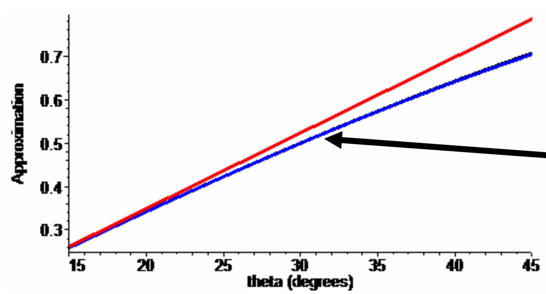
This equation cannot be solved easily unless the *small angle approximation* is used…

How small is a small angle?

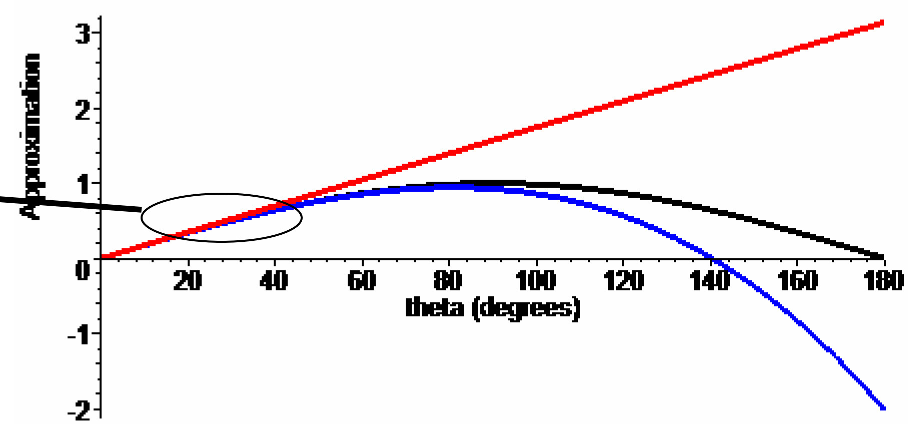
- The small angle approximation is a very important result in many areas of Physics and Applied Mathematics
  - The *Taylor Series* expansion yields:

$$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} -,+,\ldots \frac{\theta^{2n-1}}{(2n-1)!}$$

- ***1st Term: Linear***
- ***2nd Term: Cubic***
- ***Actual: Sine***



Approximation nearly perfect for angles less than 20 degrees…



Percent Error in the Small Angle Approximation…

In the **small-angle approximation** we assumed that $\theta << 1$ and used the approximation $\sin\theta \cong \theta$. We are now going to decide what is a "**small**" angle, i.e., up to what angle θ is the approximation reasonably accurate?

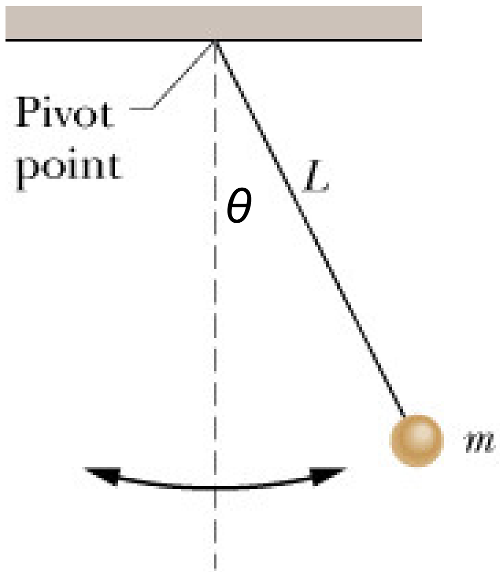
θ (degrees)	**θ (radians)**	**$\sin\theta$**
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259 (1% off)
20	0.349	0.342 (2% off)

Conclusion: <u>If we keep $\theta < 10°$ we make less than 1 % error</u>.

The Simple Pendulum as SHO

- **Begin with Newton's second law *(rotational form)***
  - $\tau = I\,\alpha$
- **Use gravitational force**
  - $\tau = -\,mg\,L\,\sin\theta$
- **Construct the differential equation**
  - $I\,d^2\theta/dt^2 + mgL\,\sin\theta = 0$
- ***Assume small angle approximation***
  - $I\,d^2\theta/dt^2 + mgL\,\theta = 0$
- **Substitute the solution** $\theta(t) = \theta_m \cos(\omega t + \varphi)$
  - $-\,\omega^2\,\theta(t) + mgL/I\,\theta(t) = 0$
- **Solve for the angular frequency**
  - $\omega^2 = mgL/I$
- **Express the period of the motion (**$I = mL^2$**)**
  - $T = 2\pi\sqrt{I/mgL} = 2\pi\sqrt{L/g}$

Simple Pendulum



Conservation of Angular Momentum

- **Angular momentum about the pivot**
  - Pure rotation so L given by:

$$L = I\omega = \left(mR^2\right)\omega$$

  - Angular velocity is given by:

$$\omega = {}^{2\pi}\!/_{T}$$

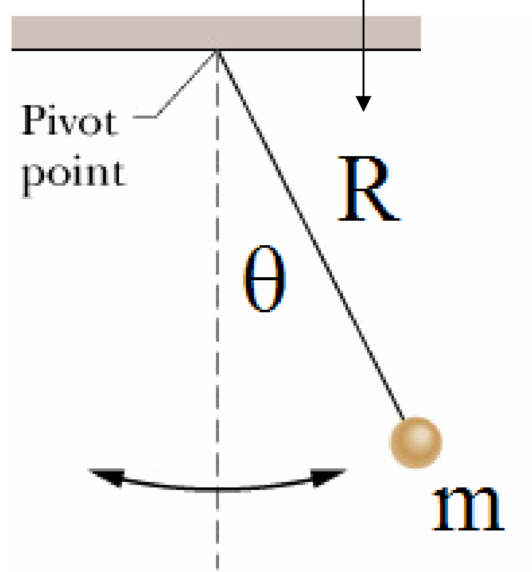
  - Period of motion is given by:

$$T = 2\pi\sqrt{{}^{R}\!/_{g}}$$

  - Angular momentum is a constant:

$$L = mR^2 \frac{2\pi}{2\pi\sqrt{{}^{R}\!/_{g}}} = m\sqrt{gR^3}\Diamond$$

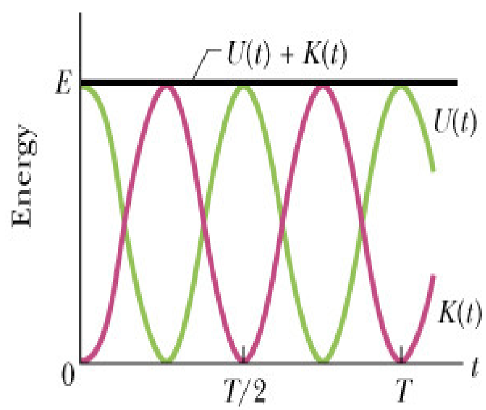
Choose R for radius so you don't get confused!



Energy in the SHO System

In the figure we plot the potential energy U (green line), the kinetic energy K (red line), and the mechanical energy E (black line) versus time t. While U and K vary with time, the energy E is a constant. The energy of the oscillating object transfers back and forth between potential and kinetic energy, while the sum of the two remains constant.

$$\omega \equiv \sqrt{\frac{k}{m}} \Rightarrow \quad \omega^2 = \frac{k}{m}$$



Potential energy: $U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$

Kinetic energy: $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi) = \frac{1}{2}m\frac{k}{m}x_m^2 \sin^2(\omega t + \phi)$

Mechanical energy: $E = U + K = \frac{1}{2}kx_m^2\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] = \frac{1}{2}kx_m^2$

$$\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) = 1$$

SHO Energy Continued…

- **The total energy in the SHO system may be derived from first principles**
  - Start with Newton's Second Law

$$\Rightarrow m\frac{dv}{dt} = -k\left(y - y_0\right)$$

$$\Rightarrow mvdv = -k\left(y - y_0\right)dy$$

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

Use chain rule "trick" here to eliminate time…

$$\Rightarrow m\int_0^v vdv = -k\int_{y_0+y_m}^{y}\left(y - y_0\right)dy$$

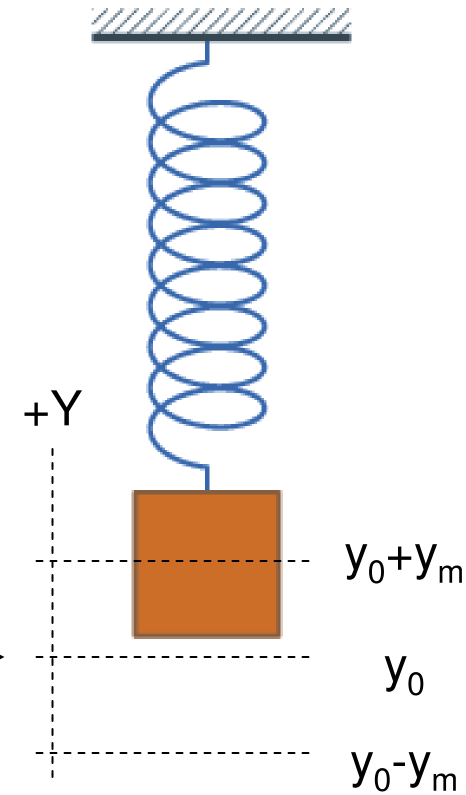
$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}ky_m^2 - \frac{1}{2}k\left(y - y_0\right)^2$$

$$\Rightarrow v(y) = \sqrt{\frac{k}{m}\left[y_m^2 - \left(y - y_0\right)^2\right]}$$

*The Maximum velocity occurs at $y = y_0$:

$$v_m = \omega y_m$$

Vertical SHO System



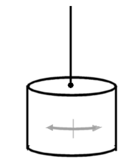
Activity – Torsion Pendulum

- Consider the torsion pendulum
  - Set down coordinates
  - Develop equations of motion
  - Determine the period of small angle oscillations

Your Name (**Print**): ______________________ Date: __________
Group Members: ______________________ Group: __________

Torsion Pendulum

1) If you hang a mass on the end of a steel wire, then twist it, and release it, it will oscillate back and forth rotationally. This is called a torsional spring, and it's a nice example of how the same basic steps in the mathematics of SHM can be applied to a large range of different problems.



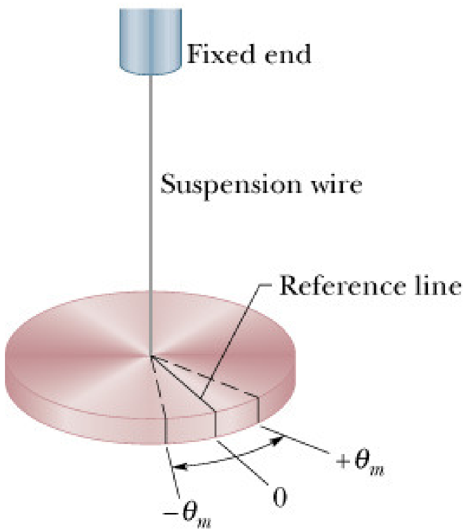
(a) State what the rotational version of Hooke's law would be for such a spring. Think about Hooke's law for a "regular" spring and replace the linear variables with rotational analogies.

(b) How would you define the spring constant for such an object? Let's call it κ ("kappa"). What units would this quantity have? On what properties of the wire do you think the torsional spring constant would depend?

(c) Apply Newton's Second Law (rotational form) and write down the resulting differential equation. (Assume the I is just some unknown moment of inertia, not necessarily a cylinder).

(d) What type of motion results from this? Write down an expression for the position angle of the object as a function of time.

The Torsion Pendulum



In the figure we show another type of oscillating system. It consists of a disc of rotational inertia I suspended from a wire that twists as m rotates by an angle θ. The wire exerts a restoring torque $\tau = -\kappa\theta$ on the disc. This is the angular form of Hooke's law. The constant κ is called the torsion constant of the wire.

$$\underbrace{I\alpha}_{\tau} = -\kappa\theta \Rightarrow \frac{d^2\theta(t)}{dt^2} + \left(\frac{\kappa}{I}\right)\theta(t) = 0$$

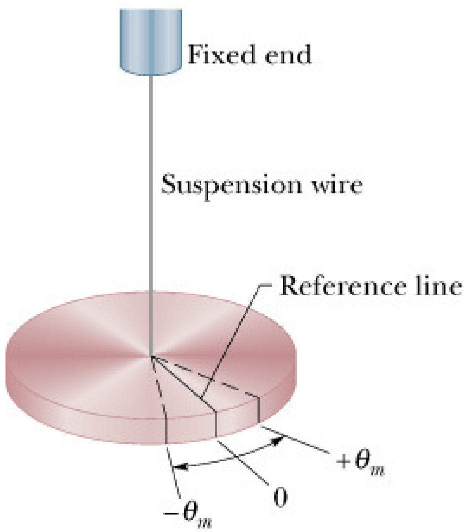
If we compare the expression $\tau = -\kappa\theta$ for the torque with the force equation $F = -Cx$ we realize that we identify the constant C with the torsion constant κ. We can thus readily determine the angular frequency ω and the period T of the oscillation:

$$\omega = \sqrt{\frac{C}{I}} = \sqrt{\frac{\kappa}{I}} \quad \Longrightarrow \quad T = 2\pi\sqrt{\frac{I}{C}} = 2\pi\sqrt{\frac{I}{\kappa}}$$

The Torsion Pendulum as SHO

- **Begin with Newton's second law** *(rotational form)*
  - $\tau = I \alpha$
- **Use Hooke's law for the force**
  - $\tau = - \kappa \theta$
- **Construct the differential equation***
  - $I \, d^2 \theta / dt^2 + \kappa \theta = 0$
- **Substitute the solution** $\theta(t) = \theta_m \cos(\omega t + \varphi)$*
  - $-\omega^2 \theta(t) + (\kappa / I) \theta(t) = 0$
- **Solve for the angular frequency**
  - $\omega^2 = \kappa / I$
- **Express the period of the motion**
  - $T = 2\pi \sqrt{I / \kappa}$

Torsion-Pendulum



*Note: Because the equation looks like the SHO equation the torsion pendulum is also an SHO!