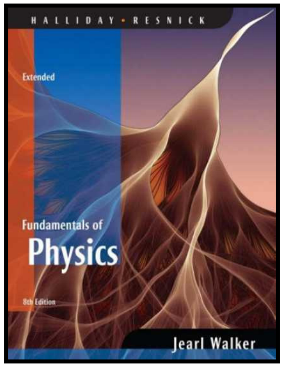
Workshop Physics 1017 - 312

# University Physics II

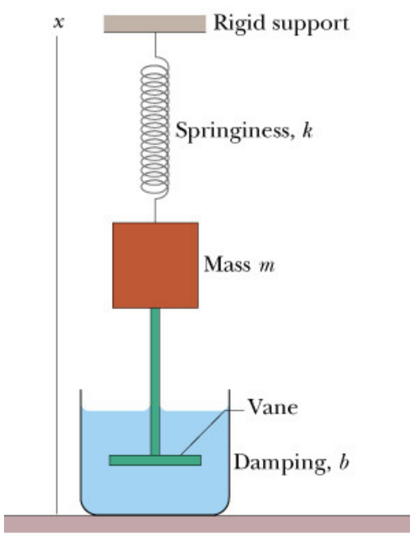
## Week 5 : Day 3



# Outline

- **Damped Harmonic Oscillator (DHO)**
  - Equations of motion
  - DHO Energy
- **Forced Harmonic Oscillator (FHO)**
  - FHO Dynamics
  - Resonance Conditions
- **Activity - Harmonic Oscillator Problems**
  - Set up for different problems
  - Compare theory with experiment

# Damped Harmonic Motion




When the amplitude of an oscillating object is reduced due to the presence of an external force the motion is said to be damped. An example is given in the figure. A mass $m$ attached to a spring of spring constant $k$ oscillates vertically. The oscillating mass is attached to a vane submerged in a liquid. The liquid exerts a damping force $\vec{F}_d$ whose magnitude is given by the equation $F_d = -bv$.

The negative sign indicates that $\vec{F}_d$ opposes the motion of the oscillating mass. The parameter $b$ is called the damping constant. The net force on m is $F_{\text{net}} = -kx - bv$. From Newton's second law we have:

What are the units of $b$?

Damped Harmonic Oscillator (DHO) Equation ⇒

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

# Solving the DHO Equation

- **Solve the DHO equation as follows**
  - First put the equation in differential form

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0 \Rightarrow \left(\frac{d^2}{dt^2}+\frac{b}{m}\frac{d}{dt}+\frac{k}{m}\right)x(t)=0$$

  - Get the characteristic equations from roots

$$\Rightarrow \left(D^2+2\gamma D+\omega^2\right)x(t)=0, \qquad D\equiv\frac{d}{dt},\ \gamma\equiv\frac{b}{2m},\ \omega=\sqrt{\frac{k}{m}}$$

  - Therefore the solution is of the form

$$\Rightarrow (D+\omega'')(D-\omega'')x(t)=0, \qquad \omega''\equiv\underbrace{-\gamma\pm\frac{1}{2}\sqrt{4\gamma^2-4\omega^2}}_{Quadratic\quad Roots}=\gamma\pm i\underbrace{\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}}_{\omega'>0}$$

$$x(t)=Ae^{-\gamma t}\left[B\cos(\omega' t)+C\sin(\omega' t)\right]$$

$$\Rightarrow x(t)=\underbrace{x_m e^{-\frac{b}{2m}t}}_{Decays}\underbrace{\cos(\omega' t+\phi)}_{Oscillates}, \qquad \omega'\equiv\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

*Note: The characteristic equations are linear DEs with solutions like:

$$(D\pm\omega'')x(t)=0$$
$$\Rightarrow x(t)=Ae^{\pm\omega'' t}$$

# DHO Simulations

- **Physlet Model of the DHO System**
  - The model is generally valid for *positive* angular frequency values, *ω'*.

$$x(t) = \underbrace{x_m e^{-\frac{b}{2m}t}}_{Decays} \underbrace{\cos(\omega' t + \phi)}_{Oscillates}, \qquad \omega' \equiv \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

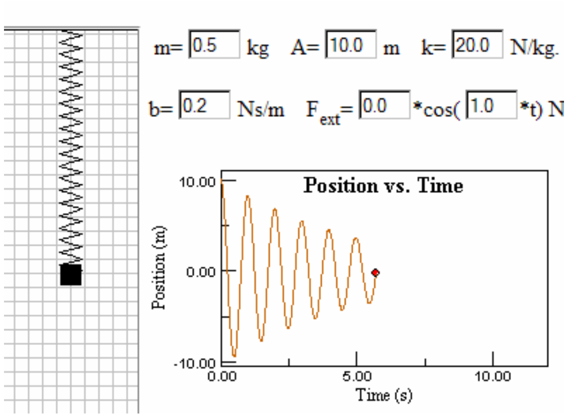
  - If the damping coefficient, $b$ is too large the system becomes *critically*- or *over- damped.*

$$\Rightarrow \frac{k}{m} - \frac{b^2}{4m^2} \leq 0$$

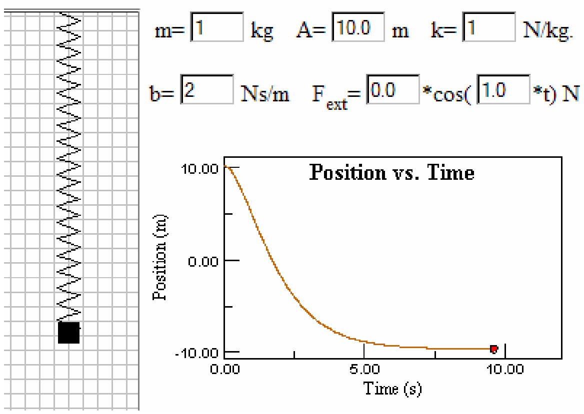
$$\Rightarrow b \geq 2\sqrt{km} \begin{cases} b = 2\sqrt{km}, & \text{Critically Damped} \\ b > 2\sqrt{km}, & \text{Over Damped} \end{cases}$$

View Physlet online at: http://www.evsis.org/applets/cm_osc.html

Under-Damped DHO Simulation

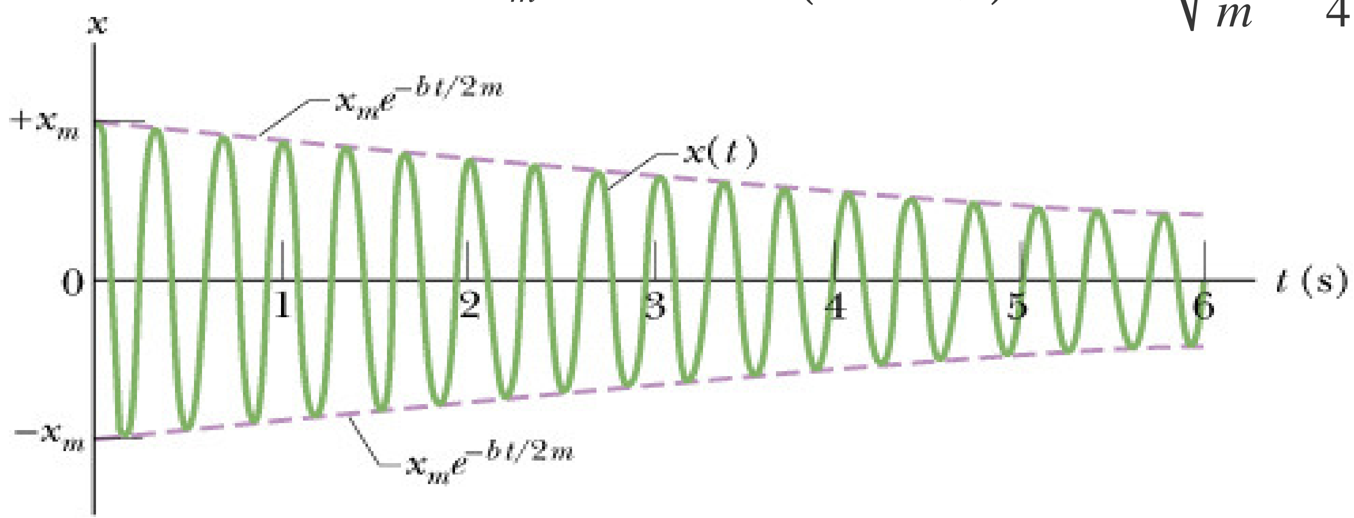



Critically-Damped DHO Simulation



# Energy in the DHO System

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$




In the picture above we plot $x(t)$ versus $t$. We can regard the above solution as a cosine function with a time-dependent amplitude $x_m e^{-bt/2m}$.

If the oscillator is damped its energy is not constant but decreases with time.

# A Closer Look at DHO Energy

- **First express total energy directly (!)**

$$E(t)=\frac{1}{2}mv^2(t)+\frac{1}{2}kx^2(t)=\frac{1}{2}m\left[\overbrace{-\frac{b}{2m}x_m e^{-\frac{b}{2m}t}\cos(\omega' t+\phi)-\omega' x_m e^{-\frac{b}{2m}t}\sin(\omega' t+\phi)}^{v(t)}\right]^2+\frac{1}{2}k\overbrace{x_m^2 e^{-\frac{b}{m}t}\cos^2(\omega' t+\phi)}^{x^2(t)}$$

  - Then simplify by factoring

$$E(t)=\frac{1}{2}m\omega'^2 x_m^2 e^{-\frac{b}{m}t}\left[\frac{b}{2m\omega'}\cos(\omega' t+\phi)+\sin(\omega' t+\phi)\right]^2+\frac{1}{2}kx_m^2 e^{-\frac{b}{m}t}\cos^2(\omega' t+\phi)$$

  - Now approximate the frequency for small damping

$$\omega'\equiv\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}=\sqrt{\frac{k}{m}\left(1-\frac{b^2}{4mk}\right)}\cong\sqrt{\frac{k}{m}},b<<2\sqrt{mk}$$

  - Therefore the energy decreases exponentially

$$E(t)\approx\frac{1}{2}m\omega'^2 x_m^2 e^{-\frac{b}{m}t}\sin^2(\omega' t+\phi)+\frac{1}{2}kx_m^2 e^{-\frac{b}{m}t}\cos^2(\omega' t+\phi)=\frac{1}{2}kx_m^2 e^{-\frac{b}{m}t}$$

# Quantifying DHO Energy Loss

- **How Long does it take for a DHO to lose a given percentage of it's energy?**
  - Use energy approximation to determine the time:
    - *Express approximate energy in terms of initial energy*

$$E(t) = \underbrace{\frac{1}{2}kx_m^2}_{E_0} e^{-\frac{b}{m}t} = E_0 e^{-\frac{b}{m}t} \Rightarrow \frac{E(t)}{E_0} = e^{-\frac{b}{m}t}$$

    - *Define a time constant* $\tau \equiv \frac{m}{b}$ *and solve for t*

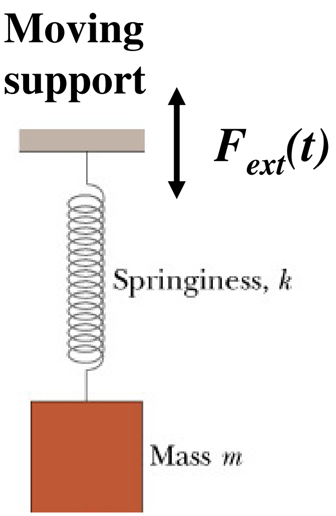
$$\frac{E(t)}{E_0} = e^{-t/\tau} \Rightarrow t = -\tau \ln\left[\frac{E(t)}{E_0}\right]$$

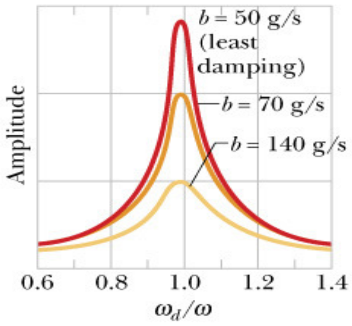
    - *Example: For b=0.2 kg/s and m=100g how long until 80%* of initial energy is lost?*

*Note: 80% lost means 20% of initial energy is left…

$$t = -\tau \ln\left[\frac{E(t)}{E_0}\right] = -\frac{0.100kg}{0.2kg/s}\ln\left[\frac{0.20E_0}{E_0}\right] = -0.5s\ln(0.20) = +0.8s$$

# Forced Harmonic Motion




If an oscillating system is disturbed and then allowed to oscillate freely the corresponding angular frequency $\omega$ is called the natural frequency. The same system can also be driven as shown in the figure by a moving support that oscillates at an arbitrary angular frequency $\omega_d$. Such a forced oscillator oscillates at the angular frequency $\omega_d$ of the driving force. The displacement is given by $x(t) = x_m \cos\left(\omega' t + \phi\right)$. The oscillation amplitude $x_m$ varies with the driving frequency as shown in the lower figure. The amplitude is approximately greatest when $\omega_d = \omega$. This condition is called resonance.

$$\frac{d^2 x(t)}{dt^2} + \frac{k}{m} x(t) = F_{ext}(t)$$



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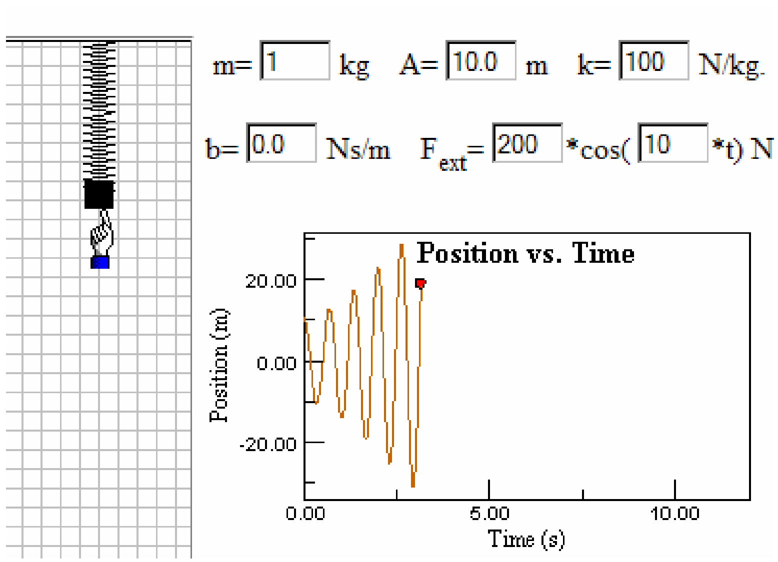
# FHO Simulations

- **Use the Physlet to see resonance**
  - Adjust driving frequency to match natural frequency for undamped system
  - See applet online at:

http://www.evsis.org/applets/cm_osc.html




$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10\frac{rad}{\sec}$$

# Activity - Sample Problems

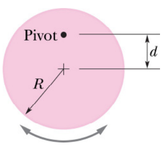
- **Problem set-up**
  - Identify force(s)
  - Generate DE
  - Find frequency
  - Express period

- **Examples**
  - Set up problems 1, 2 for extra practice…

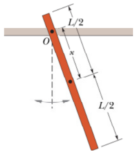
Your Name (**Print**): ____________________ Date: ________
Group Members: ____________________ Group: ________
____________________

**Simple Harmonic Motion Problems**

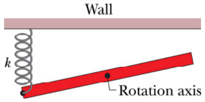
1. *{HRW8 15.P.045}* A physical pendulum consists of a uniform solid disk of radius $R$ supported in a vertical plane by a pivot located a distance $d$ from the center of the disk. The disk is displaced a small angle and released.
   a) Derive an expression for the period of the resulting simple harmonic motion in terms of the given quantities.
   b) Calculate a numerical value for the period if $R = 2.35$ cm and $d = 1.75$ cm.



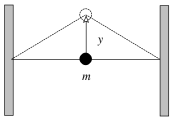

2. *{HRW8 15.P.049}* A stick of length $L$ oscillates as a physical pendulum.
   a) Derive an expression for the period as a function of the distance $x$ between the pivot point and the center of mass of the stick.
   b) Derive an expression for the value of $x$ that gives the <u>shortest</u> period.
   c) Calculate a numerical value for the shortest period if $L = 1.85$ m.




3. *{HRW8 15.P.051}* In the overhead view shown in the diagram, a long uniform rod of length $L$ and mass $M$ is free to rotate in a horizontal plane about a vertical axis through its center. A spring with force constant $k$ is connected horizontally between one end of the rod and a fixed wall. When the rod is in equilibrium, it is parallel to the wall.
   a) When displaced slightly from equilibrium, is the motion simple harmonic motion? Why?
   b) If the motion is simple harmonic, derive an expression for the period in terms of the given quantities.



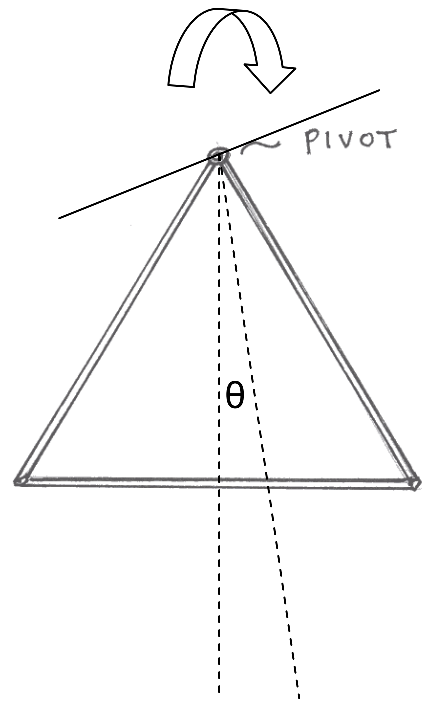

4. A mass $m$ is connected to two identical rubber bands of length $L$, each under tension $T$ as shown in the diagram. The mass moves on a smooth horizontal surface. Assume the magnitude of the tension in the rubber bands does not change when the mass is displaced a small distance $y$ from the equilibrium position.
   a) Does the mass execute simple harmonic motion when it is displaced a small distance $y$ and released? Why?
   b) If the motion is simple harmonic, derive an expression for the period in terms of the given quantities.



# Physical Pendulum Period

- Theory and Experiment (Problem 5)
  - a) Calculate the period of a physical pendulum formed by joining three identical meter sticks into a triangle as shown and letting it pivot through a small angle about one of the corners.
  - b) Measure the period using the assembled triangle in the room. Does your calculated value agree with the measured value within a reasonable uncertainty? If not, repeat part a)!
- Testing your prediction
  - Use the solution you obtain and build a model using meter sticks and tape
  - Each table should first agree upon the right answer and then make and test their own model

Balance pendulum on a knife-edge support




Consider small angle oscillations and time ...illation





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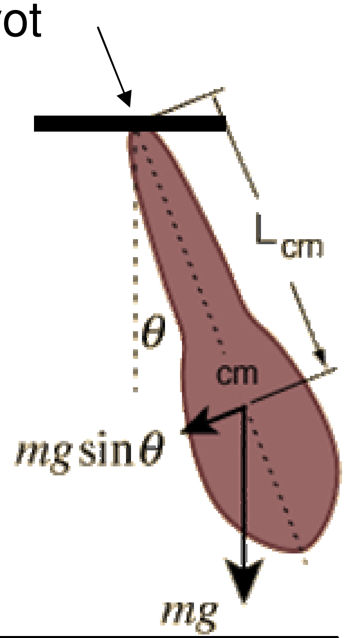
# Physical Pendulum Model

- Hanging objects may be made to oscillate
- Motion described by *Newton's 2nd Law*
  - Use rotational form

Picture credits Hyperphysics site at: http://hyperphysics.phy-astr.gsu.edu/Hbase/pendp.html




$$\tau = I_{object}\alpha$$

$$-L_{cm}(mg\sin\theta) = I_{object}\alpha$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{mgL_{cm}}{I_{object}}\sin\theta = 0$$

$$sin\theta \approx \theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{mgL_{cm}}{I_{object}}\theta = 0$$

$$\Rightarrow \theta(t) = \theta_0 \sin\left(\sqrt{\frac{mgL_{cm}}{I_{object}}}t\right)$$

The *small angle approximation*, gives solution as SHO system so that the frequency and period of oscillation may be predicted:

$$\Rightarrow \omega = \sqrt{\frac{mgL_{cm}}{I_{object}}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{I_{object}}{mgL_{cm}}}$$

# Center of Mass for System

- **Where is the COM?**
  - Recall that the COM for point masses is given by:

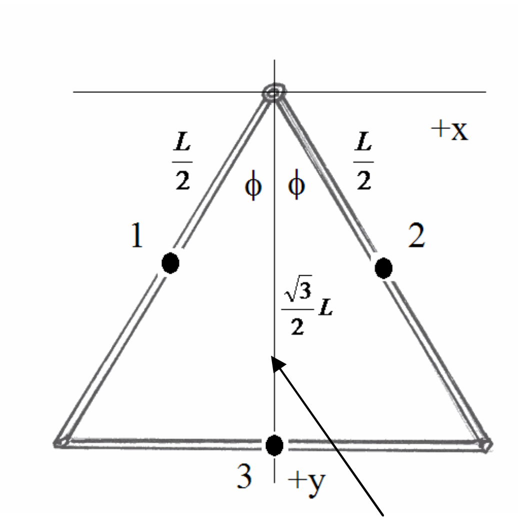
$$\vec{r}_{com} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_{i_i}}$$

  - The location is then given by

$$x_{com} = 0,\ y_{com} = \frac{my_1 + my + my_3}{3m}$$

$$\Rightarrow \frac{1}{3}\left[\frac{L}{2}\cos\phi + \frac{L}{2}\cos\phi + \frac{\sqrt{3}}{2}L\right]$$

$$\Rightarrow \frac{1}{3}\left[\frac{L}{2}\frac{\sqrt{3}}{2} + \frac{L}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}L\right] = \frac{\sqrt{3}}{3}L$$




COM is located 2/3 of height…

$$\Rightarrow L_{cm} = \frac{\sqrt{3}}{3}L = \frac{2}{3}\left(\frac{\sqrt{3}}{2}L\right)$$