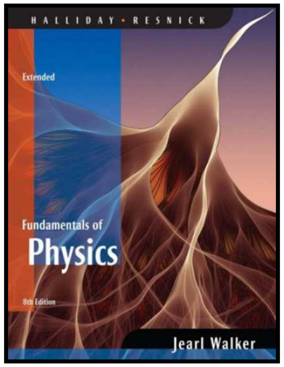
Workshop Physics 1017 - 312

University Physics II



Week 6 : Day 3

Outline

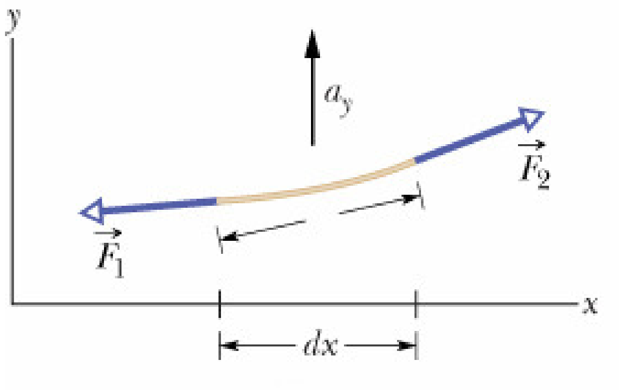
- ☐ **The Wave equation**
  - ➢ Wave speed of a wave
  - ➢ Wave Distribution and Energy
    - *Energy transmission along a stretched string*

- ☐ **Standing waves**
  - ➢ Formation of a Standing Wave
  - ➢ The Standing Wave Model
    - *Nodes and Antinodes*

  - ➢ Resonance in Standing waves
    - *The Resonance conditions*

- ☐ **Workshop Activity - Applets**
  - ➢ Superposition, Phasors and Standing Waves

The Wave Equation

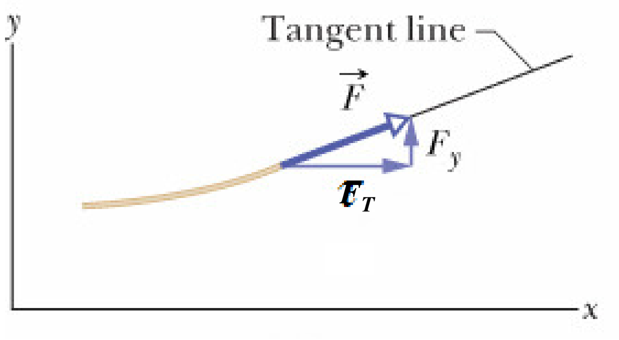


Consider a string of mass density μ and tension τ.

A transverse wave propagates along the string.

The transverse motion is described by $y(x,t)$.

Consider an element of length dx and mass $dm = \mu dx$.



$$\Rightarrow \frac{F_y}{F_T} = \frac{dy}{dx}$$

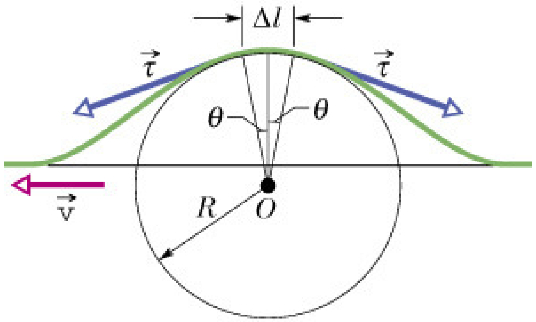
From Newton's second law we have:

$$dF_y = dm a_y \Rightarrow \frac{d}{dx} F_y = \frac{dm}{dx}\left(\frac{d^2 y}{dt^2}\right)$$

$$\Rightarrow \frac{d}{dx}\left(F_T \frac{dy}{dx}\right) = \mu \frac{d^2 y}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2 y}{dx^2} = \frac{\mu}{F_T} \frac{d^2 y}{dt^2}}$$

Wave Speed on a Stretched String



$$\Rightarrow \frac{1}{v^2} = \frac{\mu}{F_T} \Rightarrow v = \boxed{\sqrt{\frac{F_T}{\mu}}}$$

From the wave equation we can find the velocity of the wave directly as follows:

$$\frac{d^2y}{dx^2} = \frac{\mu}{F_T}\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{dy}{dt}\frac{dt}{dx} = \frac{1}{v}\frac{d}{dx}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{v}\frac{dt}{dx}\frac{d}{dt}\frac{dy}{dt} = \frac{1}{v^2}\frac{d^2y}{dt^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$$

Note : The speed v depends on the tension τ and the mass density μ but not on the wave frequency f.

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Wave Distribution & Energy

- **The Wave Disturbance**
  - May visualize using 3D plots

$$y(x,t) = y_m \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi\right)$$

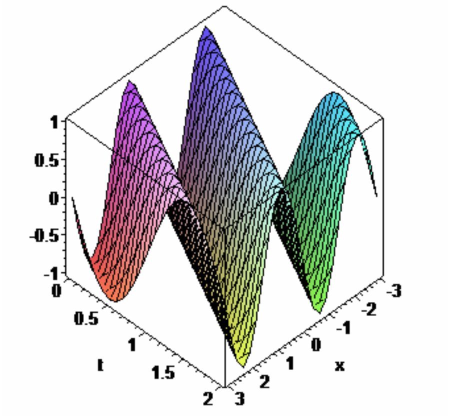
  - Consider and $\lambda = 3\ m$ and $T = 2\ s$
- **The Wave Energy**
  - Depends upon transverse velocity, v_T

$$v_T(x,t) \equiv \frac{d}{dt}y(x,t) = -\frac{2\pi}{T}y_m \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi\right)$$

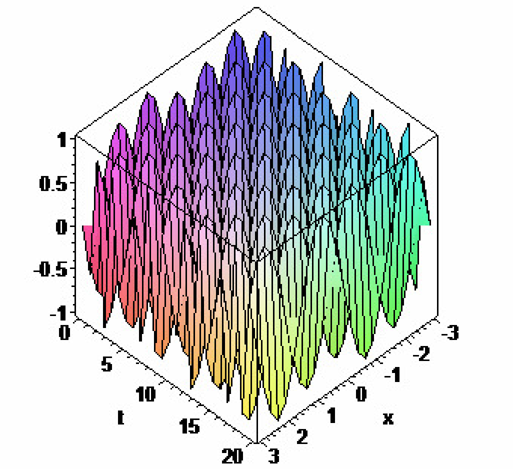
  - Depends upon amplitude squared

$$E(x,t) = \frac{1}{2}mv_T^2(x,t) = \frac{1}{2}m\omega^2 y_m^2 \cos^2(kx - \omega t + \phi)$$

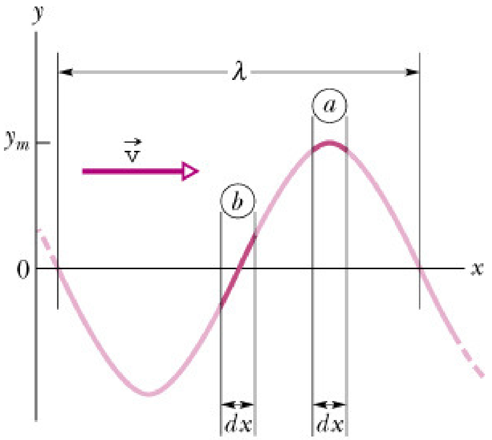
One Period



Ten Periods



Rate of Energy Transmission



Consider a transverse wave propagating along a string, which is described by the equation $y(x,t) = y_m \sin(kx - \omega t)$. The transverse velocity $u = \dfrac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$. At point a both y and u are equal to zero. At point b both y and u have maxima.

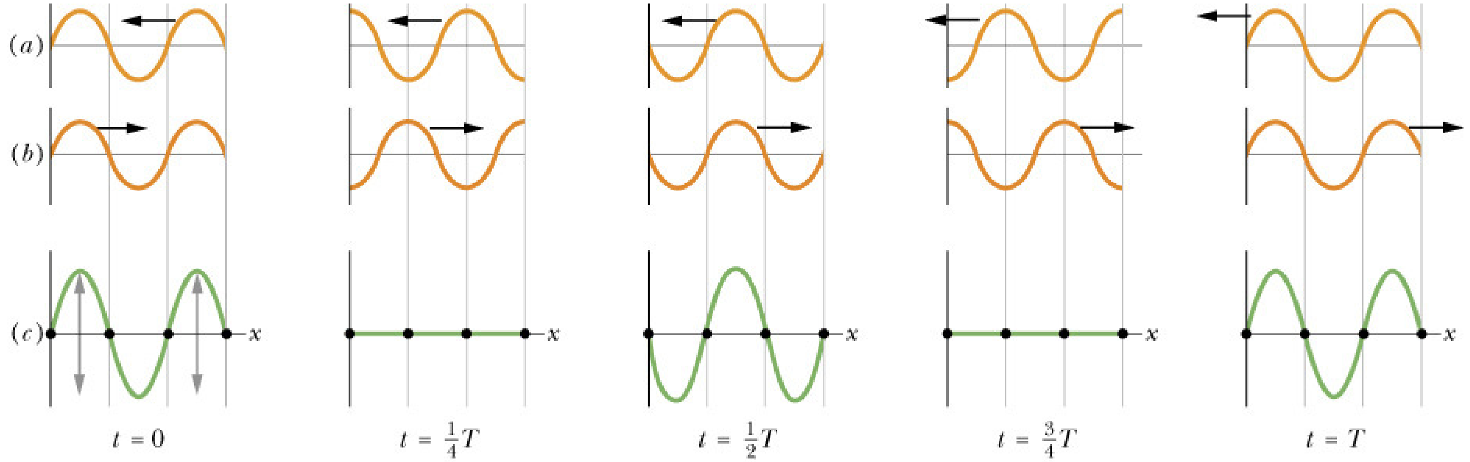
$$\left(\frac{dK}{dt}\right)_{\text{avg}} = \frac{1}{2}\mu v \omega^2 y_m^2 \left[\cos^2(kx - \omega t)\right]_{\text{avg}} = \frac{1}{4}\mu v \omega^2 y_m^2.$$

As in the case of the oscillating spring-mass system,

$$\left(\frac{dU}{dt}\right)_{\text{avg}} = \left(\frac{dK}{dt}\right)_{\text{avg}} \rightarrow P_{\text{avg}} = \left(\frac{dU}{dt}\right)_{\text{avg}} + \left(\frac{dK}{dt}\right)_{\text{avg}} \boxed{= \frac{1}{2}\mu v \omega^2 y_m^2.}$$

$$\mu v \Rightarrow \frac{dm}{dx}\frac{dx}{dt}$$

Standing Wave Formation



Standing Waves : Consider the superposition of two waves that have the same frequency and amplitude but travel in opposite directions. The displacements of two waves are $y_1(x,t) = y_m \sin(kx - \omega t)$ and $y_2(x,t) = y_m \sin(kx + \omega t)$.

The displacement of the resulting wave $y'(x,t) = y_1(x,t) + y_2(x,t)$

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) = \boxed{[2y_m \sin kx]\cos \omega t.}$$

This is not a traveling wave but an oscillation that has a position-dependent amplitude. It is known as a ***standing wave.***

Adapted from Halliday, Resnick and Walker; © Michael Jay Schillaci

Standing Wave Model

- ❑ **Consider two waves traveling in opposite directions:**

$$y'(x,t) = \underbrace{y_m \sin(kx-\omega t)}_{traveling \quad right} + \underbrace{y_m \sin(kx+\omega t)}_{traveling \quad left}$$

- ❑ **Use double angle formulae:**

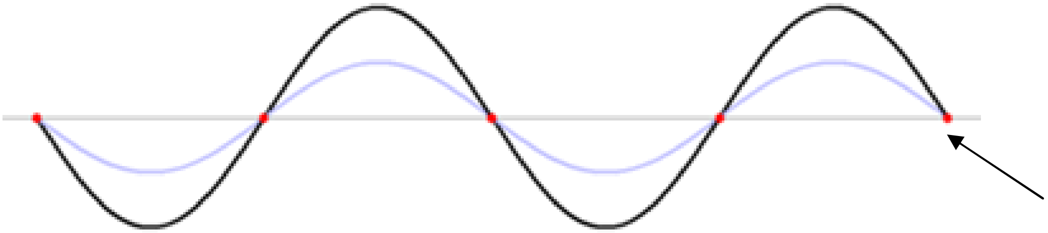
$$\sin(kx-\omega t) = \sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)$$
$$\sin(kx+\omega t) = \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)$$

- ❑ **Add the resulting waves:**

$$y'(x,t) = 2y_m \sin(kx)\cos(\omega t)$$

Note: This is not a harmonic wave!

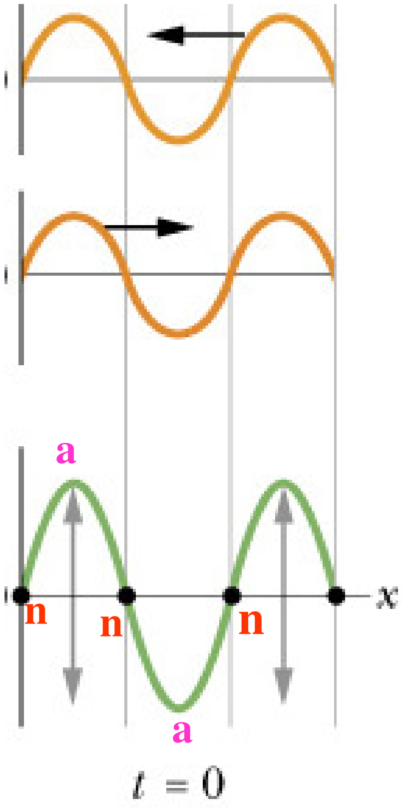
A Standing Wave is a sum of two harmonic waves traveling in opposite directions…



The stationary points or "nodes" of a Standing wave do not move

Picture credit: http://en.wikipedia.org/wiki/Standing_wave

Nodes and Antinodes



The displacement of a standing wave is given by the equation

$y'(x,t) = [2y_m \sin kx]\cos \omega t.$

The position-dependent amplitude is equal to $2y_m \sin kx.$

Nodes. These are defined as positions where the standing wave amplitude vanishes. They occur when $kx = n\pi$ for $n = 0,1,2,$

$$\rightarrow \frac{2\pi}{\lambda}x = n\pi \rightarrow x_n = n\frac{\lambda}{2} \quad \text{for} \quad n = 0,1,2,....$$

Antinodes. These are defined as positions where the standing wave amplitude is maximum.

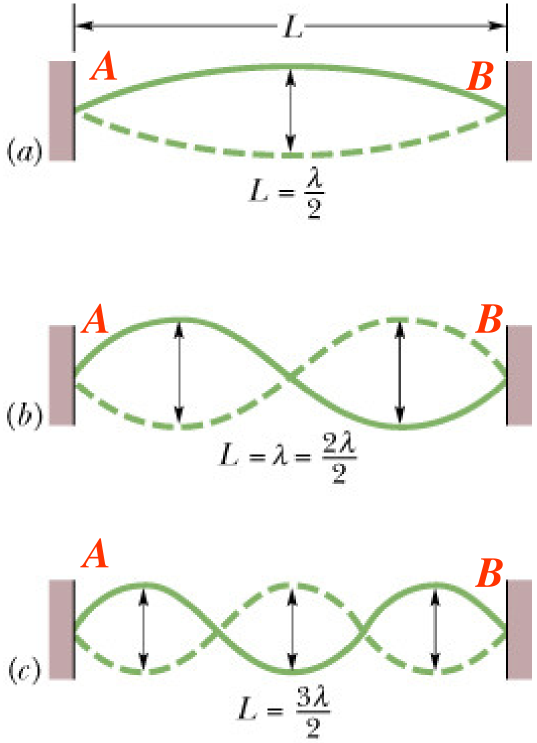
They occur when $kx = \left(n + \frac{1}{2}\right)\pi$ for $n = 0,1,2,....$

$$\rightarrow \frac{2\pi}{\lambda}x = \left(n + \frac{1}{2}\right)\pi \rightarrow x'_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \quad \text{for} \quad n = 0,1,2,....$$

Note 1: The distance between adjacent nodes and antinodes is $\lambda/2$.

Note 2: The distance between a node and an adjacent antinode is $\lambda/4$.

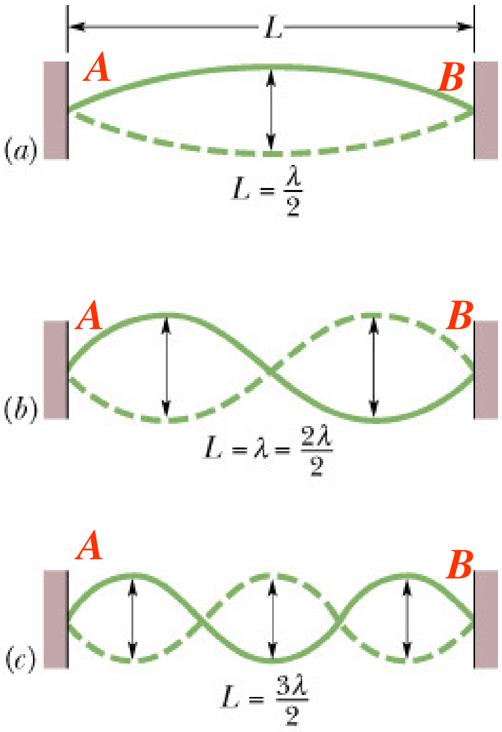
Resonance in Standing Waves



Consider a string under tension that is clamped at points A and B separated by a distance L. We send a harmonic wave traveling to the right. The wave is reflected at point B and the reflected wave travels to the left. The left-going wave reflects back at point A and creates a thrird wave traveling to the right. Thus we have a large number of overlapping waves, half of which travel to the right and the rest to the left.

For certain frequencies the interference produces a standing wave. Such a standing wave is said to be **at resonance**. The frequencies at which the standing wave occurs are known as the **resonant frequencies** of the system.

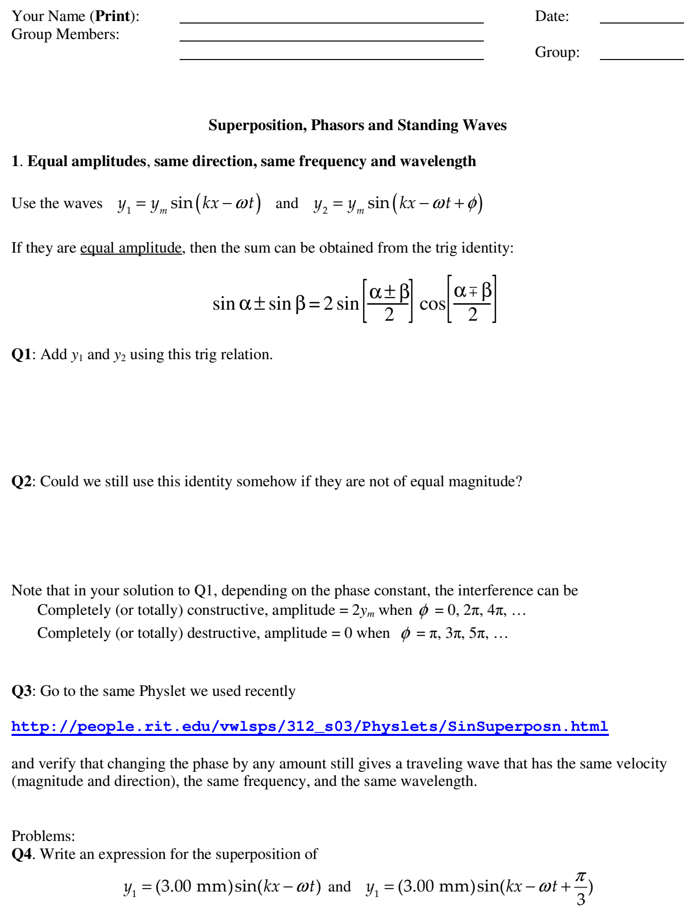
Resonance conditions



Resonances occur when the resulting standing wave satisfies the boundary condition of the problem. These are that the amplitude must be zero at point A and point B and arise from the fact that the string is clamped at both points and therefore cannot move. The first resonance is shown in fig. a. The standing wave has two nodes at points A and B. Thus $L = \frac{\lambda_1}{2}$ $\rightarrow \lambda_1 = 2L$. The second standing wave is shown in fig. b. It has three nodes (two of them at A and B). In this case $L = 2\left(\frac{\lambda}{2}\right) = \lambda \rightarrow \lambda_2 = L$. The third standing wave is shown in fig. c. It has four nodes (two of them at A and B). In this case $L = 3\left(\frac{\lambda}{2}\right) = \lambda \rightarrow \lambda_3 = \frac{2}{3}L$. The general expression for the resonant wavelengths is $\lambda_n = \frac{2L}{n}$ for $n = 1, 2, 3, \ldots$. The resonant frequencies $f_n = \frac{v}{\lambda_n} = n\frac{v}{2L}$.

Activity – Superposition…

- ❑ **Superposition & Phasors**
  - ➢ Adding two or more waves using trigonometric relationships
  - ➢ Adding two or more waves using phasor diagrams
- ❑ **Standing waves**
  - ➢ Conditions for a standing wave
  - ➢ Finding nodes and antinodes



Your Name (**Print**): ______________________ Date: ________
Group Members: ______________________
______________________ Group: ________

Superposition, Phasors and Standing Waves

1. Equal amplitudes, same direction, same frequency and wavelength

Use the waves $y_1 = y_m \sin\left(kx - \omega t\right)$ and $y_2 = y_m \sin\left(kx - \omega t + \phi\right)$

If they are equal amplitude, then the sum can be obtained from the trig identity:

$$\sin\alpha \pm \sin\beta = 2\sin\left[\frac{\alpha \pm \beta}{2}\right]\cos\left[\frac{\alpha \mp \beta}{2}\right]$$

Q1: Add y_1 and y_2 using this trig relation.

Q2: Could we still use this identity somehow if they are not of equal magnitude?

Note that in your solution to Q1, depending on the phase constant, the interference can be
Completely (or totally) constructive, amplitude = $2y_m$ when $\phi = 0, 2\pi, 4\pi, \ldots$
Completely (or totally) destructive, amplitude = 0 when $\phi = \pi, 3\pi, 5\pi, \ldots$

Q3: Go to the same Physlet we used recently

http://people.rit.edu/vwlsps/312_s03/Physlets/SinSuperposn.html

and verify that changing the phase by any amount still gives a traveling wave that has the same velocity (magnitude and direction), the same frequency, and the same wavelength.

Problems:
Q4. Write an expression for the superposition of

$$y_1 = (3.00\text{ mm})\sin(kx - \omega t) \text{ and } y_1 = (3.00\text{ mm})\sin(kx - \omega t + \frac{\pi}{3})$$